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TESTS FOR STRUCTURAL CHANGE IN COINTEGRATED SYSTEMS

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This paper considers tests for structural change of the cointegrating vector and the adjustment vector in the error correction model with an unknown change point. This paper derives new tests for structural change, which are applicable to maximum likelihood estimation. Our tests for structural change of the cointegrating vector have the same nonstandard asymptotic distributions that have been found by Hansen (1992a, *Journal of Business and Economic Statistics* 10, 321–335). In contrast, the tests on the adjustment vector have the same asymptotic distributions that have been found by Andrews and Ploberger (1994, *Econometrica* 62, 1383–1414) for models with stationary variables. Asymptotic critical values are provided.

1. INTRODUCTION

This paper considers tests for structural change of the cointegrating vector and the adjustment vector in the error correction model (ECM) with an unknown change point. The purpose is to develop the appropriate test statistics and the associated distribution theory in cointegrated systems. It is of interest and use because many economic studies have questioned the stability of long-run equilibrium relationships. Particularly, there is vast literature on the stability of the money demand equation, including Lucas (1988) and Stock and Watson (1993).

The stability of long-run relationships can be statistically assessed by testing structural change of the cointegrating vector between the variables. Some tests of this form have been proposed by Hansen (1992a) and Quintos and Phillips (1993), which use the fully modified estimator of the cointegrating vector. These tests are not applicable to maximum likelihood estimation and thereby exclude most potential applications. This paper fills this gap in the literature by deriving new tests based on the maximum likelihood estimator (MLE) from the ECM.

The distribution theory of the cointegrating vector in the Gaussian ECM has been developed by Johansen (1988, 1991). We use the same model, but we assume that the cointegrating vector can be identified with a normalization condition. We define Lagrange multiplier (LM) statistics for structural change in the

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cointegrating vector and the adjustment vector by using the efficient score. Because the LM statistics do not require sequential estimation, our tests are computationally easy and fast.

Conventional LM statistics are defined with respect to a known break point, but we relax this constraint by allowing an unknown break point. In this case, classical optimality theory does not hold because a nuisance parameter exists only under the alternative hypothesis. To deal with this difficulty, alternative testing procedures have been suggested by Davies (1977, 1987), King and Shively (1993), Andrews (1993), and Andrews and Ploberger (1994). We define average (Ave-LM), exponential average (Exp-LM), and supremum (Sup-LM) LM statistics based on the optimality arguments of Andrews (1993) and Andrews and Ploberger (1994).

This paper finds that the tests for structural change of the cointegrating vector have the same nonstandard asymptotic distributions that have been found by Hansen (1992a), although Hansen (1992a) used the fully modified estimator. In contrast, the tests on the adjustment vector have the same asymptotic distributions that have been found by Andrews and Ploberger (1994) for models with stationary variables.

This paper also extends the stability tests to models with deterministic trends. We consider three models: (1) no drift, (2) no trend in the data-generating process (DGP), and (3) trend in the DGP. Asymptotic critical values of Ave-LM, Exp-LM, and Sup-LM tests are provided for each model.

There are other related papers by Hansen and Johansen (1993), Quintos (1993), and Quintos (1997). Hansen and Johansen (1993) and Quintos (1993) considered the likelihood ratio method for detecting structural change by recursive estimation of the cointegration space. Quintos (1997) developed the parameter stability tests using the Wald criterion. All these authors use sequential estimation methods; hence, they are complementary to the result of this paper.

We denote $\rightarrow^p$ as convergence in probability and $\Rightarrow$ as weak convergence with respect to the uniform metric. The expression $B(s) = BM(\Omega)$ represents a Brownian motion with long-run variance Ω. Also, $[\cdot]$ is the integer operator, $\operatorname{tr} A$ is the trace of matrix A, and $\operatorname{vec}(\cdot)$ is the column-stacking operator.

The next section explains the model and defines the LM statistics for structural change of the cointegrating vector and the adjustment vector. Section 3 explores the asymptotic distribution theory for these tests. Models with deterministic trends are considered in Section 4. Section 5 deals with simulation results of the asymptotic critical values. Small sample Monte Carlo experiments are also done to find the power and the size distortion of our tests. An empirical application to the money demand equation is made in Section 6.

2. THE MODEL AND PRELIMINARY RESULTS

Consider a p-dimensional time series x_t generated by the ECM that allows one-time structural change in the cointegrating vector as follows:

$$\Delta x_t = \alpha \begin{pmatrix} I_r \\ \beta + \delta\{t \geq [n\tau] + 1\} \end{pmatrix}' x_{t-1} + \sum_{i=1}^{l-1} \Gamma_i \Delta x_{t-i} + u_t, \tag{1}$$

where α is a $p \times r$ full column rank matrix, β and δ are $(p - r) \times r$ matrices, $\{\cdot\}$ is the indicator function, and u_t is independent and identically distributed (i.i.d.) with mean zero and covariance matrix Σ.

We assume that the cointegration rank is known and equals r. Therefore, if we denote equation (1) as $\Pi(L)x_t = u_t$, then the rank of $\Pi(= -\Pi(1))$ is r.

Our model implicitly assumes a normalization condition of the cointegration space. According to our normalization, x_t can be partitioned into r-dimensional x_{1t} and $(p - r)$-dimensional x_{2t}. This normalization is a special case of representing the ECM. From this representation, the cointegrating vector can be identified. The same normalization was used by Phillips (1991) in his triangular representation. In principle, any ordering of x_{1t}, x_{2t} may be possible if the partitioned matrix of cointegrating vectors corresponding to x_{1t} is nonsingular and x_{2t} is not itself cointegrated. In some cases we can specify the normalization according to economic theory. For example, in a money demand equation, if we set x_{1t} to be real balances and x_{2t} to be real income and the nominal interest rate, then we can interpret the cointegrating vector β as the long-run income elasticity and the interest semielasticity of money demand.

We define the cointegrating relationship (or the long-run relationship) as

$$w_t = x_{1t} + (\beta + \delta\{t \geq [n\tau] + 1\})' x_{2t}, \tag{2}$$

which is stationary (or $I(0)$).

Our definition of the cointegrating relationship is different from that of Engle and Granger (1987) because we allow one-time structural change in the cointegrating vector at the break point τ. This is new and unconventional. Our model can be reduced to the conventional model if the cointegrating vector is stable; therefore, the conventional model is a special case of our model.

The break point τ intersects two subsamples, $t = 1,2,\ldots,[n\tau]$ and $t = [n\tau] + 1,\ldots,n$. Hence, the corresponding cointegrating vectors are β and $\beta + \delta$, respectively. We treat τ as fixed until we define optimal tests for unknown τ.

The null and alternative hypotheses for the stability of the cointegrating vector β are

$$\mathcal{H}_0^\beta: \delta = 0 \quad \text{and} \quad \mathcal{H}_1^\beta: \delta \neq 0.$$

Assumption 1.

(a) $\tau \in \tau^*$ and $\tau^* = [\underline{\tau}, \bar{\tau}] \subset (0,1)$.
(b) $\{u_t\} \sim$ i.i.d. $(0,\Sigma)$.

We define the parameter vector

$$\theta = \text{vec}(\delta, \beta, \alpha, \Gamma, \Sigma) \in \Theta,$$

where

$$\Gamma = (\Gamma_1', \Gamma_2', \ldots, \Gamma_{l-1}')'.$$

We also define the σ-field $\mathcal{F}_t$ generated by x_{t-i} for $i = 1,2,\ldots$. The log-likelihood function, with the auxiliary condition that u_t is normally distributed, is given by

$$\mathcal{L}_n(\theta,\tau) = \sum_{t=1}^{[n\tau]} l_t(0,\beta,\alpha,\Gamma,\Sigma) + \sum_{t=[n\tau]+1}^{n} l_t(\delta,\beta,\alpha,\Gamma,\Sigma), \tag{3}$$

where

$$l_t(\theta) = -\tfrac{1}{2}\log|\Sigma| - \tfrac{1}{2}\operatorname{tr} u_t(\theta)u_t'(\theta)\Sigma^{-1},$$

and $u_t = u_t(\theta)$ in equation (1).

We denote $\hat{\theta}(\tau)(=\hat{\theta}_n(\tau))$ as the unrestricted MLE of θ for known $\tau \in \tau^*$. That is,

$$\hat{\theta}(\tau) = \underset{\theta\in\Theta}{\operatorname{argmax}}\, \mathcal{L}_n(\theta,\tau). \tag{4}$$

If we denote $\tilde{\theta}(=\tilde{\theta}_n)$ as the restricted MLE of θ, then

$$\tilde{\theta} = \underset{\theta\in\Theta,\delta=0}{\operatorname{argmax}}\, \mathcal{L}_n(\theta,\tau). \tag{5}$$

Our tests use the restricted MLE. Methods to compute the restricted MLE $\tilde{\theta}$ have been suggested by Ahn and Reinsel (1988) and Box and Tiao (1977).

The first-order conditions of the unrestricted likelihood function are given by

$$\frac{\partial \mathcal{L}_n(\hat{\theta},\tau)}{\partial \delta} = \sum_{t=[n\tau]+1}^{n} x_{2t-1}\hat{u}_t'\hat{\Sigma}^{-1}\hat{\alpha} = 0, \tag{6}$$

$$\frac{\partial \mathcal{L}_n(\hat{\theta},\tau)}{\partial \beta} = \sum_{t=1}^{n} x_{2t-1}\hat{u}_t'\hat{\Sigma}^{-1}\hat{\alpha} = 0, \tag{7}$$

$$\frac{\partial \mathcal{L}_n(\hat{\theta},\tau)}{\partial \alpha'} = \sum_{t=1}^{n} \hat{w}_{t-1}\hat{u}_t'\hat{\Sigma}^{-1} = 0, \tag{8}$$

and

$$\frac{\partial \mathcal{L}_n(\hat{\theta},\tau)}{\partial \Gamma_i'} = \sum_{t=1}^{n} \Delta x_{t-i}\hat{u}_t'\hat{\Sigma}^{-1} = 0, \quad \text{for } i = 1,2,\ldots,l-1, \tag{9}$$

where $\hat{u}_t = u_t(\hat{\theta})$ in equation (1) and $\hat{w}_t = x_{1t} + (\hat{\beta} + \hat{\delta}\{t \geq [n\tau]+1\})'x_{2t}$.

We denote $\tilde{u}_t = u_t(\tilde{\theta})$ in equation (1), and $z_t = (\Delta x_t', \Delta x_{t-1}', \ldots, \Delta x_{t-l+2}')'$.

Let

$$\lambda_n^{\beta}(\tau) = \left(\tilde{\alpha}'\tilde{\Sigma}^{-1} \otimes n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)\tilde{u}_t'\right)\operatorname{vec}(I),$$

where

$$R_{12t}(\tau) = x_{2t-1} - \sum_{t=1}^{[n\tau]} x_{2t-1}z_{t-1}'\left(\sum_{t=1}^{[n\tau]} z_{t-1}z_{t-1}'\right)^{-1} z_{t-1}.$$

The restricted MLE $\tilde{\theta}$ satisfies the first-order conditions except equation (6). We call $\lambda_n^{\beta}(\tau)$ the Lagrange multiplier (or the score), which is based on equation (6) and an asymptotically negligible term. The score function uses the partial sum, for example, $\sum_{t=1}^{[n\tau]} z_{t-1}z_{t-1}'$ instead of the grand sum because the former can be properly extended to the model with deterministic trends.

We define the LM statistic for the null hypothesis $\mathcal{H}_0^{\beta}$ as follows:

$$\mathrm{LM}_n^{\beta}(\tau) = \lambda_n^{\beta'}(\tau)[\text{Est. Var}(\lambda_n^{\beta}(\tau))]^{-1}\lambda_n^{\beta}(\tau).$$

If we use the asymptotic results in Section 3, we have the following:

$$\lambda_n^{\beta}(\tau) = \left(\alpha'\Sigma^{-1} \otimes n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)u_t'\right)\mathrm{vec}(I) - \left(\alpha'\Sigma^{-1} \otimes S_{11n}(\tau)S_{11n}(1)^{-1}n^{-1}\sum_{t=1}^{n} R_{12t}(1)u_t'\right)\mathrm{vec}(I) + o_{p\tau}(1),$$

where $S_{11n}(\tau) = n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)R_{12t}(\tau)'$ and $\sup_{\tau\in\mathcal{T}^*} o_{p\tau}(1) = o_p(1)$.

Thus, the estimated variance of the score $\lambda_n^{\beta}(\tau)$ is given by

$$\text{Est. Var}(\lambda_n^{\beta}(\tau)) = \tilde{\alpha}'\tilde{\Sigma}^{-1}\tilde{\alpha} \otimes V_{11n}(\tau),$$

where $V_{11n}(\tau) = n^{-1}S_{11n}(\tau) - n^{-1}S_{11n}(\tau)S_{11n}(1)^{-1}S_{11n}(\tau)$ and $\tilde{\Sigma} = n^{-1}\sum_{t=1}^{n}\tilde{u}_t\tilde{u}_t'$.

Let

$$g_n^{\beta}(\tilde{\theta},\tau) = n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)\tilde{v}_t', \tag{10}$$

where $\tilde{v}_t = (\tilde{\alpha}'\tilde{\Sigma}^{-1}\tilde{\alpha})^{-1/2'}\tilde{\alpha}'\tilde{\Sigma}^{-1}\tilde{u}_t$.

If we neglect asymptotically negligible terms, the LM statistic for $\mathcal{H}_0^{\beta}$ against $\mathcal{H}_1^{\beta}$ is given by

$$\mathrm{LM}_n^{\beta}(\tau) = \mathrm{tr}\, g_n^{\beta'}(\tilde{\theta},\tau)[V_{11n}(\tau)]^{-1}g_n^{\beta}(\tilde{\theta},\tau). \tag{11}$$

The LM statistic is a simple function of the data and the restricted MLE $\tilde{\theta}$. Because we can use existing estimation methods, it is computationally easy and fast. In contrast, Wald statistics and likelihood ratio statistics for this model require sequential estimation, which is much more computationally burdensome.

2.1. Stability of α

Consider the tests for structural change of the adjustment vector α for a known break point τ in the equation

$$\Delta x_t = (\alpha + \epsilon\{t \geq [n\tau] + 1\})\begin{pmatrix} I \\ \beta \end{pmatrix}' x_{t-1} + \sum_{i=1}^{l-1}\Gamma_i\Delta x_{t-i} + u_t. \tag{12}$$

In this case, the null and the alternative hypotheses are

$$\mathcal{H}_0^{\alpha}\colon \epsilon = 0 \quad \text{and} \quad \mathcal{H}_1^{\alpha}\colon \epsilon \neq 0.$$

The parameter vector θ is now defined as $\theta = \mathrm{vec}(\epsilon, \beta, \alpha, \Gamma, \Sigma)$, and the log-likelihood function is given by

$$\mathcal{L}_n^\alpha(\theta,\tau) = \sum_{t=1}^{[n\tau]} l_t(0,\beta,\alpha,\Gamma,\Sigma) + \sum_{t=[n\tau]+1}^{n} l_t(\epsilon,\beta,\alpha,\Gamma,\Sigma),$$

where

$$l_t(\theta) = -\tfrac{1}{2}\log|\Sigma| - \tfrac{1}{2}\operatorname{tr} u_t(\theta)u_t'(\theta)\Sigma^{-1},$$

and $u_t = u_t(\theta)$ in equation (12).

Note that we have the same restricted MLE $\tilde{\theta}$ as before. The first-order conditions of the unrestricted likelihood function are given by equations (7)–(9), and

$$\frac{\partial \mathcal{L}_n(\hat{\theta},\tau)}{\partial \epsilon'} = \sum_{t=[n\tau]+1}^{n} \hat{w}_{t-1}\hat{u}_t'\hat{\Sigma}^{-1} = 0. \tag{13}$$

The restricted MLE $\tilde{\theta}$ satisfies equations (7)–(9), but it does not satisfy equation (13). We have the following Lagrange multiplier, which is based on equation (13) and an asymptotically negligible term:

$$\lambda_n^\alpha(\tau) = \left(\tilde{\Sigma}^{-1} \otimes n^{-1/2}\sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau)\tilde{u}_t'\right)\mathrm{vec}(I),$$

where

$$\tilde{R}_{2t}(\tau) = \tilde{w}_{t-1} - \sum_{t=1}^{[n\tau]} \tilde{w}_{t-1}z_{t-1}'\left(\sum_{t=1}^{[n\tau]} z_{t-1}z_{t-1}'\right)^{-1} z_{t-1}, \quad \text{and} \quad \tilde{w}_t = x_{1t} + \tilde{\beta}'x_{2t}.$$

We define the LM statistic for the null hypothesis $\mathcal{H}_0^\alpha$ as follows:

$$\mathrm{LM}_n^\alpha(\tau) = \lambda_n^{\alpha\prime}(\tau)[\text{Est. Var}(\lambda_n^\alpha(\tau))]^{-1}\lambda_n^\alpha(\tau).$$

To estimate the variance of the score $\lambda_n^\alpha(\tau)$, we use the following asymptotic result:

$$\begin{aligned} &n^{-1/2}\sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau)\tilde{u}_t' \\ &\quad = n^{-1/2}\sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau)u_t' - S_{22n}(\tau)S_{22n}(1)^{-1}n^{-1/2}\sum_{t=1}^{n} \tilde{R}_{2t}(1)u_t' + o_{p\tau}(1), \end{aligned}$$

where $S_{22n}(\tau) = n^{-1}\sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau)\tilde{R}_{2t}(\tau)'$.

The estimated variance of the score $\lambda_n^\alpha(\tau)$ is given by

$$\text{Est. Var}(\lambda_n^\alpha(\tau)) = \tilde{\Sigma}^{-1} \otimes V_{22n}(\tau),$$

where $V_{22n}(\tau) = S_{22n}(\tau) - S_{22n}(\tau)S_{22n}(1)^{-1}S_{22n}(\tau)$.

Let

$$g_n^\alpha(\tilde{\theta},\tau) = n^{-1/2}\sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau)\tilde{u}_t'\tilde{\Sigma}^{-1/2}. \tag{14}$$

The LM statistic of the tests for $\mathcal{H}_0^\alpha$ against $\mathcal{H}_1^\alpha$ is given by

$$\mathrm{LM}_n^\alpha(\tau) = \operatorname{tr} g_n^{\alpha\prime}(\tilde{\theta},\tau)[V_{22n}(\tau)]^{-1} g_n^\alpha(\tilde{\theta},\tau). \tag{15}$$

2.2. Joint Stability of β, α

Now, we consider the tests of the one-time structural change in β and α for a known break point τ in the equation

$$\Delta x_t = (\alpha + \epsilon\{t \geq [n\tau] + 1\})\begin{pmatrix} 1 \\ \beta + \delta\{t \geq [n\tau] + 1\} \end{pmatrix}' x_{t-1} + \sum_{i=1}^{l-1} \Gamma_i \Delta x_{t-i} + u_t. \tag{16}$$

The null and the alternative hypotheses to test for structural change of β and α jointly are

$$\mathcal{H}_0^{\beta\alpha}\colon \delta = \epsilon = 0 \quad \text{and} \quad \mathcal{H}_1^{\beta\alpha}\colon \delta \neq 0 \text{ or } \epsilon \neq 0.$$

In this case, the parameter vector is defined $\theta = \operatorname{vec}(\delta,\epsilon,\beta,\alpha,\Gamma,\Sigma)$ and the restricted MLE $\tilde{\theta}$ can be defined as before.

We define the LM statistic for the null hypothesis $\mathcal{H}_0^{\beta\alpha}$ as follows:

$$\lambda_n^{\beta\alpha}(\tau) = \begin{pmatrix} \lambda_n^\beta(\tau) \\ \lambda_n^\alpha(\tau) \end{pmatrix}.$$

To calculate the asymptotic variance of the score $\lambda_n^{\beta\alpha}(\tau)$, we use the following asymptotic result:

$$\begin{pmatrix} \lambda_n^\beta(\tau) \\ \lambda_n^\alpha(\tau) \end{pmatrix} = \begin{pmatrix} \left(\alpha'\Sigma^{-1} \otimes n^{-1} \sum_{t=1}^{[n\tau]} R_{12t}(\tau) u_t'\right) \operatorname{vec}(I) \\ \left(\Sigma^{-1} \otimes n^{-1/2} \sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau) u_t'\right) \operatorname{vec}(I) \end{pmatrix}$$

$$- \begin{pmatrix} \left(\alpha'\Sigma^{-1} \otimes S_{11n}(\tau)(S_{11n}(1))^{-1} n^{-1} \sum_{t=1}^{n} R_{12t}(1) u_t'\right) \operatorname{vec}(I) \\ \left(\Sigma^{-1} \otimes S_{22n}(\tau)(S_{22n}(1))^{-1} n^{-1/2} \sum_{t=1}^{n} \tilde{R}_{2t}(1) u_t'\right) \operatorname{vec}(I) \end{pmatrix}$$

$$+ o_{p\tau}(1).$$

From the asymptotic results in Section 3, $n^{-1/2} S_{12n}(\tau) \to^p 0$ uniformly in $\tau \in \tau^*$, where $S_{12n}(\tau) = n^{-1} \sum_{t=1}^{[n\tau]} R_{12t}(\tau) \tilde{R}_{2t}(\tau)'$. This result implies that the variance of the score $\lambda_n^{\beta\alpha}(\tau)$ is block-diagonal asymptotically. Thus, the LM statistic of $\mathcal{H}_0^{\beta\alpha}$ against $\mathcal{H}_1^{\beta\alpha}$ is asymptotically equivalent to the simple sum

$$\mathrm{LM}_n^{\beta\alpha}(\tau) = \mathrm{LM}_n^{\beta}(\tau) + \mathrm{LM}_n^{\alpha}(\tau). \tag{17}$$

2.3. Unknown Break Point

We have defined the LM statistics for fixed τ. This is appropriate when τ is known to the econometrician. More typically in applied work, it is natural that the break point τ is thought to be unknown. In this case, the testing procedure is nonstandard because a nuisance parameter τ appears only under the alternative hypothesis. Tests with specific optimality properties have been proposed by Davies (1977, 1987), King and Shively (1993), Andrews (1993), and Andrews and Ploberger (1994). We follow Andrews (1993) and Andrews and Ploberger (1994), whose method is based on the weighted power criterion function with respect to the randomized nuisance parameter.

If we assume that τ lies in $\tau^* = [\underline{\tau}, \bar{\tau}]$, then the optimal tests are defined as follows:

$$\text{Ave-LM}_n^i = \frac{1}{\bar{t} - \underline{t}} \sum_{t=\underline{t}}^{\bar{t}} \mathrm{LM}_n^i([t/n]),$$

$$\text{Exp-LM}_n^i = \log\left(\frac{1}{\bar{t} - \underline{t}} \sum_{t=\underline{t}}^{\bar{t}} \exp(\mathrm{LM}_n^i([t/n])/2)\right),$$

and

$$\text{Sup-LM}_n^i = \operatorname*{Max}_{t\in[\underline{t},\bar{t}]} \mathrm{LM}_n^i([t/n]),$$

where $\underline{t} = [n\underline{\tau}]$, $\bar{t} = [n\bar{\tau}]$, and $i = \beta, \alpha, \beta\alpha$.

Andrews and Ploberger (1994) proposed the asymptotically optimal test statistic that is a function of the concentration parameter c. Because the Ave-LM statistic is defined with respect to $c \to 0$, the power is concentrated on the alternative that is near the null hypothesis. The Exp-LM statistic is defined with respect to $c \to \infty$, and so the power is concentrated on the alternative that is very far from the null hypothesis. The power of the Sup-LM statistic is also concentrated on the distant alternative hypothesis.

3. MAIN RESULTS

We use the following representation theorem (for proof, see Engle and Granger, 1987; Johansen, 1991).

THEOREM 1. (Granger–Johansen representation) *Assume that the null hypotheses are valid, so that there is no structural change. We assume that x_t is integrated of order one under the null hypotheses. Suppose the cointegration* *relation holds, that is,* $\Pi = \alpha\gamma'$, *where* $\gamma = (I, \beta')'$, *and* α *and* γ *are* $p \times r$ *full column rank matrices. If* $\alpha_\perp$ *and* $\gamma_\perp$ *are* $p \times (p-r)$ *full column rank matrices such that* $\alpha_\perp'\alpha = 0$ *and* $\gamma_\perp'\gamma = 0$, *then the ECM can be represented by*

(1)

$$\Delta x_t = C(L)u_t,$$

with $C(1) = \gamma_\perp(\alpha_\perp'\Psi(1)\gamma_\perp)^{-1}\alpha_\perp'$, *where* $\Psi(1)$ *is the derivative of* $\Pi(L)$ *for* $L = 1$,

(2)

$$x_t = C(1)\sum_{i=1}^{t} u_i + \Phi(L)u_t,$$

where $\Phi(L) = (C(L) - C(1))/(1-L)$, *and*

(3)

$$w_t = \gamma' x_t = \gamma'\Phi(L)u_t.$$

Under the null hypotheses, the representation theorem implies that x_t can be decomposed into stochastic trends and a stationary component. The cointegration matrix γ eliminates the stochastic trends; hence, the cointegrating relationship is stationary.

Typically, we define $\Rightarrow$ as weak convergence on the space $C[0,1]$ with respect to the uniform metric. Here, we need to define weak convergence of the projected sequence $R_{12t}(\tau) = x_{2t-1} - \sum_{t=1}^{[n\tau]} x_{2t-1} z_{t-1}' (\sum_{t=1}^{[n\tau]} z_{t-1} z_{t-1}')^{-1} z_{t-1}$. Hence, we need $\Rightarrow$ to denote weak convergence on $C[0,1] \otimes \tau^*$ with respect to the uniform metric $\rho(\cdot,\cdot)$, where

$$\rho(g,h) = \sup_{s\in[0,1],\tau\in\tau^*} |g(s,\tau) - h(s,\tau)|,$$

and $|\cdot|$ is a matrix norm. That is, $|A| = (\operatorname{tr} A'A)^{1/2}$.

Assumption 2.

(c) $E|u_0|^2 < \infty$.
(d) $\sum_{k=1}^{\infty} k^2 |C_k| < \infty$, where $C(L) = \sum_{k=0}^{\infty} C_k L^k$
(e) $\sup_{\theta\in\Theta} |\theta| < \infty$.

We denote $C_2(1)$ as a partitioned matrix of $C(1)$ that corresponds to x_{2t}; hence its dimension is $(p-r) \times p$.

LEMMA 1. *Under the null hypotheses and Assumptions* 1 *and* 2,

$$n^{-1/2}\sum_{t=1}^{[ns]} u_t \Rightarrow W(s), \tag{18}$$

$$n^{-1/2}\sum_{t=1}^{[ns]} w_t \Rightarrow \gamma'\Phi(1)W(s), \tag{19}$$

and

$$n^{-1/2}R_{12[ns]}(\tau) \Rightarrow C_2(1)W(s) \equiv W_2(s), \tag{20}$$

where $W(s) = BM(\Sigma)$.

To show the main theorem, we use a weak convergence theorem of Hansen (1992b).

LEMMA 2. *Under the null hypotheses of no structural change and Assumptions* 1 *and* 2,

$$n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)u_t' \Rightarrow \int_0^\tau W_2(s)\,dW'(s), \tag{21}$$

$$n^{-2}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)R_{12t}'(\tau) \Rightarrow \int_0^\tau W_2(s)W_2'(s)\,ds, \tag{22}$$

and

$$n^{-1}\sum_{t=1}^{[n\tau]} R_{2t}(\tau)R_{2t}'(\tau) \to^p \tau Q \tag{23}$$

uniformly in $\tau \in \tau^*$, *where*

$$R_{2t}(\tau) = w_{t-1} - \sum_{t=1}^{[n\tau]} w_{t-1}z_{t-1}'\left(\sum_{t=1}^{[n\tau]} z_{t-1}z_{t-1}'\right)^{-1} z_{t-1},$$

and

$$Q = E(w_0w_0') - E(w_0z_0')(E(z_0z_0'))^{-1}E(z_0w_0').$$

To simplify the main theorem, we define the following standard Brownian motions:

$$\begin{pmatrix} B_1(s) \\ B_2(s) \end{pmatrix} = \begin{pmatrix} (\alpha'\Sigma^{-1}\alpha)^{-1/2'}\alpha'\Sigma^{-1}W(s) \\ (C_2'(1)\Sigma C_2(1))^{-1/2'}C_2(1)W(s) \end{pmatrix} = BM\begin{pmatrix} I_r & 0 \\ 0 & I_{p-r} \end{pmatrix}.$$

The tests for structural change of the cointegrating vector β have the following property.

THEOREM 2. *Under* $\mathcal{H}_0^\beta$ *and Assumptions* 1 *and* 2,

$$\mathrm{LM}_n^\beta(\tau) \Rightarrow \mathrm{tr}\, F(\tau)^{b'}[V(\tau) - V(\tau)V(1)^{-1}V(\tau)]^{-1}F(\tau)^b \equiv \mathrm{LM}_1^\beta(\tau), \tag{24}$$

where

$$F(\tau)^b = F(\tau) - V(\tau)V(1)^{-1}F(1),$$

$$F(\tau) = \int_0^\tau B_2(s)\,dB_1'(s),$$

and

$$V(\tau) = \int_0^\tau B_2(s)B_2'(s)\,ds.$$

Hence,

$$\text{Ave-LM}_n^\beta \Rightarrow \frac{1}{\bar{\tau} - \underline{\tau}} \int_{\tau \in \mathcal{T}^*} \text{LM}_1^\beta(\tau)\,d\tau,$$

$$\text{Exp-LM}_n^\beta \Rightarrow \log\left(\frac{1}{\bar{\tau} - \underline{\tau}} \int_{\tau \in \mathcal{T}^*} \exp(\text{LM}_1^\beta(\tau)/2)\,d\tau\right),$$

and

$$\text{Sup-LM}_n^\beta \Rightarrow \operatorname*{Max}_{\tau \in \mathcal{T}^*} \text{LM}_1^\beta(\tau).$$

Even though vec $F(\tau)$ is distributed as mixed normal with covariance matrix $I \otimes V(\tau)$, $F(\tau)^b$ is not a Brownian bridge, as defined in the stationary case, although it is still tied down. Hence, our asymptotic distributions are different from those found by Andrews (1993) and Andrews and Ploberger (1994).

Because the distribution of LM_1^β is chi-squared for a known τ, our tests for structural change of the cointegrating vector are standard only if we know the change point. Our distribution is the same as that found by Hansen (1992a), although Hansen (1992a) used the fully modified estimator. To estimate efficient scores, bias correction is needed because the least-squares estimator of the cointegrating vector is not median-unbiased. However, this bias does not exist in our model, which makes nonparametric estimation avoidable.

The distribution of our tests depends only on the number of parameters and the admissible range of the break point. The empirical distribution and the associated asymptotic critical values can be generated by simulation. These results are presented in Section 5.

On the other hand, our tests for structural change of the adjustment vector have the same asymptotic distribution as in the stationary case, as we now show. We define $J(s)$ as a pr-dimensional standard Brownian motion, which is independent of $B_1(s)$ and $B_2(s)$.

THEOREM 3. *Under $\mathcal{H}_0^\alpha$ and Assumptions* 1 *and* 2,

$$\text{LM}_n^\alpha(\tau) \Rightarrow \frac{1}{\tau(1-\tau)} J(\tau)^{b'} J(\tau)^b \equiv \text{LM}^\alpha(\tau), \tag{25}$$

where $J(\tau)^b = J(\tau) - \tau J(1)$.

Also,

$$\text{Ave-LM}_n^\alpha \Rightarrow \frac{1}{\bar{\tau} - \underline{\tau}} \int_{\tau \in \tau^*} \text{LM}^\alpha(\tau)\, d\tau,$$

$$\text{Exp-LM}_n^\alpha \Rightarrow \log\left(\frac{1}{\bar{\tau} - \underline{\tau}} \int_{\tau \in \tau^*} \exp(\text{LM}^\alpha(\tau)/2)\, d\tau \right),$$

and

$$\text{Sup-LM}_n^\alpha \Rightarrow \operatorname*{Max}_{\tau \in \tau^*} \text{LM}^\alpha(\tau).$$

Here $J(\tau)^b$ is a standard Brownian bridge; hence, the distributions of tests for structural change of the adjustment vector α are the same as those for stability tests in stationary regressions. The distribution of the Sup-LM test has been found by Andrews (1993), and those of the Ave-LM and the Exp-LM tests have been found by Andrews and Ploberger (1994). Thus, tests for structural change of the adjustment vector can be based on the empirical critical values provided in those papers.

The joint test for structural change of β and α has the following asymptotic distribution.

COROLLARY 1. *Under* $\mathcal{H}_0^{\beta\alpha}$ *and Assumptions* 1 *and* 2,

$$\text{LM}_n^{\beta\alpha}(\tau) \Rightarrow \text{LM}_1^\beta(\tau) + \text{LM}^\alpha(\tau) \equiv \text{LM}_1^{\beta\alpha}(\tau). \tag{26}$$

Hence,

$$\text{Ave-LM}_n^{\beta\alpha} \Rightarrow \frac{1}{\bar{\tau} - \underline{\tau}} \int_{\tau \in \tau^*} \text{LM}_1^{\beta\alpha}(\tau)\, d\tau,$$

$$\text{Exp-LM}_n^{\beta\alpha} \Rightarrow \log\left(\frac{1}{\bar{\tau} - \underline{\tau}} \int_{\tau \in \tau^*} \exp(\text{LM}_1^{\beta\alpha}(\tau)/2)\, d\tau \right),$$

and

$$\text{Sup-LM}_n^{\beta\alpha} \Rightarrow \operatorname*{Max}_{\tau \in \tau^*} \text{LM}_1^{\beta\alpha}(\tau).$$

The proof comes from Theorems 2 and 3.

4. MODELS WITH DETERMINISTIC TRENDS

When a model has deterministic trends, the distribution theory changes depending on the detrending method. If we use demeaned or detrended data, then we should use the corresponding distribution theory. This section considers two models: (1) no trend in the DGP and (2) trend in the DGP, although these two models use the same ECM with drift. This section considers only tests for structural change of the cointegrating vector. The distribution theory for structural change of the adjustment vector is invariant to the detrending method in the sense that the detrending method only removes the deterministic trends. In contrast, the detrending method affects the stochastic trends of the integrated variables. The distribution theory for joint stability is straightforward.

4.1. No Trend in the DGP

Suppose we have the following model that allows one-time structural change of the cointegrating vector for a known break point τ:

$$\Delta x_t = \mu + \alpha \begin{pmatrix} I \\ \beta + \delta\{t \geq [n\tau] + 1\} \end{pmatrix}' x_{t-1} + \sum_{i=1}^{l-1} \Gamma_i \Delta x_{t-i} + u_t. \tag{27}$$

We assume in this subsection that $\alpha_\perp' \mu = 0$, which means that there is no trend in the DGP. If we denote $z_t^* = (1, z_t)$, then we have the following property.

LEMMA 3.

$$n^{-1/2} R_{12[ns]}^*(\tau) \Rightarrow W_2(s) - \frac{1}{\tau} \int_0^\tau W_2(s)\, ds \equiv W_2^*(s,\tau), \tag{28}$$

where $R_{12t}^*(\tau) = x_{2t-1} - \sum_{t=1}^{[n\tau]} x_{2t-1} z_{t-1}^{*\prime} (\sum_{t=1}^{[n\tau]} z_{t-1}^* z_{t-1}^{*\prime})^{-1} z_{t-1}^*$.

This paper uses the score function based on the partial sum, which works properly in the model with deterministic trends. On the other hand, the score function based on the grand sum entails a nuisance term, and the variance estimator of the score function does not correspond to that derived from the Hessian matrix in the model with deterministic trends.

Whereas demeaned Brownian motions are typically defined with respect to the grand mean $\int_0^1 W_2(s)\, ds$, we use the partial mean process $1/\tau \int_0^\tau W_2(s)\, ds$. Thus, $W_2^*(s,\tau)$ is an array with argument (s,τ), with $s \leq \tau$. For notational economy, we denote $W_2^*(\tau) = W_2^*(s,\tau)$. The following lemma can be verified analogously to that of Lemma 2 if we use the weak convergence theorem by Hansen (1992b).

LEMMA 4. *Under* $\mathcal{H}_0^\beta$, $\alpha_\perp' \mu = 0$, *and Assumptions* 1 *and* 2,

$$n^{-1} \sum_{t=1}^{[n\tau]} R_{12t}^*(\tau) u_t' \Rightarrow \int_0^\tau W_2^*(\tau)\, dW'(s),$$

and

$$n^{-2} \sum_{t=1}^{[n\tau]} R_{12t}^*(\tau) R_{12t}^{*\prime}(\tau) \Rightarrow \int_0^\tau W_2^*(\tau) W_2^{*\prime}(\tau)\, ds.$$

Under the null hypothesis of no structural change, the model can be estimated by existing methods. Suppose $(\tilde{\mu}, \tilde{\alpha}, \tilde{\beta}, \tilde{\Gamma}, \tilde{\Sigma})$ is estimated by those methods. We denote $\tilde{u}_t = u_t(\tilde{\mu}, \tilde{\alpha}, \tilde{\beta}, \tilde{\Gamma})$ for u_t in equation (27). We also denote $S_{11n}^*(\tau) = n^{-1} \sum_{t=1}^{[n\tau]} R_{12t}^*(\tau) R_{12t}^{*\prime}(\tau)$ and $V_{11n}^*(\tau) = n^{-1} S_{11n}^*(\tau) - n^{-1} S_{11n}^*(\tau)$ $S_{11n}^*(1)^{-1} S_{11n}^*(\tau)$.

The LM statistic of the tests for structural change of the cointegrating vector in this model can be defined

$$\mathrm{LM}_n^{\beta *}(\tau) = \operatorname{tr} g_n^{\beta *\prime}(\tilde{\theta},\tau)[V_{11n}^*(\tau)]^{-1} g_n^{\beta *}(\tilde{\theta},\tau), \tag{29}$$

where $g_n^{\beta *}(\tilde{\theta},\tau) = n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}^*(\tau)\tilde{v}_t'$ and $\tilde{v}_t = (\tilde{\alpha}'\tilde{\Sigma}^{-1}\tilde{\alpha})^{-1/2'}\tilde{\alpha}'\tilde{\Sigma}^{-1}\tilde{u}_t$.

LEMMA 5. *Under* $\mathcal{H}_0^{\beta}$, $\alpha_\perp' \mu = 0$, *and Assumptions* 1 *and* 2,

$$\mathrm{LM}_n^{\beta *}(\tau) \Rightarrow \operatorname{tr} F^*(\tau)^{b\prime}[V^*(\tau) - V^*(\tau)V^*(1)^{-1}V^*(\tau)]^{-1}F^*(\tau)^b \equiv \mathrm{LM}_2^{\beta}(\tau), \tag{30}$$

where

$$F^*(\tau)^b = F^*(\tau) - V^*(\tau)V^*(1)^{-1}F^*(1),$$

$$F^*(\tau) = \int_0^{\tau} B_2^*(s,\tau)\, dB_1(s),$$

$$V^*(\tau) = \int_0^{\tau} B_2^*(s,\tau)B_2^{*\prime}(s,\tau)\, ds,$$

and

$$B_2^*(s,\tau) = B_2(s) - \frac{1}{\tau}\int_0^{\tau} B_2(s)\, ds.$$

Hence,

$$\text{Ave-LM}_n^{\beta *} \Rightarrow \frac{1}{\bar{\tau} - \underline{\tau}}\int_{\tau\in\mathcal{T}^*} \mathrm{LM}_2^{\beta}(\tau)\, d\tau,$$

$$\text{Exp-LM}_n^{\beta *} \Rightarrow \log\left(\frac{1}{\bar{\tau} - \underline{\tau}}\int_{\tau\in\mathcal{T}^*} \exp(\mathrm{LM}_2^{\beta}(\tau)/2)\, d\tau\right),$$

and

$$\text{Sup-LM}_n^{\beta *} \Rightarrow \operatorname*{Max}_{\tau\in\mathcal{T}^*} \mathrm{LM}_2^{\beta}(\tau).$$

The proof is analogous to that of Theorem 2. Its distribution is based on the array of demeaned Brownian motions with respect to the partial mean process.

4.2. Trend in the DGP

If $\alpha_\perp' \mu$ is nonzero in equation (27), the DGP contains a linear trend. Because the linear trend cannot be removed by demeaning only, the linear trend remains and dominates stochastic trends. Hence, its distribution is different from that of no trend in the DGP.

LEMMA 6. *Suppose* $(p-r)\times(p-r-1)$ *matrix* $\nu_\perp$ *and* $(p-r)$*-dimensional vector* ν *satisfy* $\nu_\perp'\nu = 0$, *where* $\nu = C_2(1)\mu$. *The sample moments have the following asymptotic properties*:

$$n^{-1}\nu_\perp' S_{11n}^*(\tau)\nu_\perp \Rightarrow \nu_\perp' C_2(1)\int_0^\tau W^*(\tau)W^{*\prime}(\tau)\,ds\,C_2'(1)\nu_\perp,$$

$$n^{-3/2}\nu_\perp' S_{11n}^*(\tau)\nu \Rightarrow \nu_\perp' C_2(1)\int_0^\tau W^*(\tau)(s-\tau/2)\,ds\,\nu'\nu,$$

and

$$n^{-2}\nu' S_{11n}^*(\tau)\nu \Rightarrow \nu'\nu\int_0^\tau (s-\tau/2)^2\,ds\,\nu'\nu.$$

We define

$$\bar{B}_2(s,\tau) = \begin{pmatrix} B_c(s) - \dfrac{1}{\tau}\displaystyle\int_0^\tau B_c(s)\,ds \\ s-\tau/2 \end{pmatrix},$$

where $B_c(s)$ is the $(p-r-1)$-dimensional standard Brownian motion.

THEOREM 4. *Under* $\mathcal{H}_0^\beta$ *and Assumptions* 1 *and* 2,

$$\mathrm{LM}_n^{\beta*}(\tau) \Rightarrow \mathrm{tr}\,\bar{F}(\tau)^{b\prime}[\bar{V}(\tau) - \bar{V}(\tau)\bar{V}(1)^{-1}\bar{V}(\tau)]^{-1}\bar{F}(\tau)^b \equiv \mathrm{LM}_3^\beta(\tau), \tag{31}$$

where

$$\bar{F}(\tau)^b = \bar{F}(\tau) - \bar{V}(\tau)\bar{V}(1)^{-1}\bar{F}(1),$$

$$\bar{F}(\tau) = \int_0^\tau \bar{B}_2(s,\tau)\,dB_1'(s),$$

and

$$\bar{V}(\tau) = \int_0^\tau \bar{B}_2(s,\tau)\bar{B}_2'(s,\tau)\,ds.$$

Hence,

$$\text{Ave-LM}_n^{\beta*} \Rightarrow \frac{1}{\bar{\tau}-\underline{\tau}}\int_{\tau\in\tau^*} \mathrm{LM}_3^\beta(\tau)\,d\tau,$$

$$\text{Exp-LM}_n^{\beta*} \Rightarrow \log\left(\frac{1}{\bar{\tau}-\underline{\tau}}\int_{\tau\in\tau^*} \exp(\mathrm{LM}_3^\beta(\tau)/2)\,d\tau\right),$$

and

$$\text{Sup-LM}_n^{\beta*} \Rightarrow \operatorname*{Max}_{\tau\in\tau^*} \mathrm{LM}_3^\beta(\tau).$$

If a linear trend exists in the DGP, one stochastic trend is replaced with a deterministic trend. When $p - r = 1$, $\bar{B}_2(s,\tau)$ is based only on the deterministic trend $s - \tau/2$. The asymptotic distribution theory is different from that of no trend in the DGP although the same model is applied. To get the exact distribution, it is important to know the DGP correctly. If we know that the DGP contains a linear trend, then we use the asymptotic critical values of the model of trend in the DGP. The hypothesis $\alpha'_\perp \mu = 0$ can be tested with the likelihood ratio method by Johansen (1991) because the model of no trend in the DGP is a nested case of the present one.

5. SIMULATION RESULTS

We have shown that our new tests for structural change have nonstandard distributions. These distributions depend on the admissible range of change point and the number of parameters. The empirical distribution tables are generated by a sample size of 1,000 and 10,000 replications. The stochastic functional of multivariate Brownian motions and its optimal test statistics are constructed by the Gauss random number generator on an IBM RISC-6000.

The admissible range of change point is symmetrically set at [0.05,0.95], [0.15,0.85], and [0.25,0.75]. As we know the range more precisely, the test has more power because it is closer to the classical test. When we have no information on the structural change, a wider range test captures any possible structural changes. Because our model generally contains many parameters, the admissible range is limited even with a parsimonious specification. Hence, the choice depends relatively on the availability of prior information and the number of usable observations.

In Table 1, the asymptotic critical values of the Ave-LM test for structural change of the cointegrating vector and joint stability are provided. Here P and r denote the number of variables and cointegration rank respectively. The admissible change points are given symmetrically. For example, $\underline{\tau} = 0.15$ indicates that $\tau \in [0,15, 0.85]$. As $\underline{\tau}$ increases, asymptotic critical values of the Ave-LM test increase. When the model of trend in the DGP is used, the critical value of LM_3^β with $p = 2$ and $r = 1$ changes from 2.48 to 2.73 as $\underline{\tau}$ moves from 0.15 to 0.25.

Asymptotic critical values of the Exp-LM test are given in Table 2, and those of the Sup-LM test are in Table 3. Asymptotic critical values of these tests generally decrease as the admissible range of the change point becomes narrow. The asymptotic critical values do not vary much among different models. The model of trend in the DGP has generally smaller Ave-LM and Exp-LM critical values. However, critical values of Sup-LM tests are higher in this model.

The number of coefficients of α is larger than that of β because of normalization. Therefore, the critical values of the tests for joint stability largely depend on those of the stability of α.

TABLE 1. Asymptotic critical values for Ave-LM$^{\beta}$, Ave-LM$^{\beta\alpha}$

$\underline{\tau}$	10%	5%	1%	10%	5%	1%	10%	5%	1%
					$p = 2, r = 1$				
		Ave-LM$_1^{\beta}$			Ave-LM$_2^{\beta}$			Ave-LM$_3^{\beta}$	
.25	2.20	2.96	4.75	2.23	2.98	4.74	2.10	2.73	4.28
.15	2.07	2.71	4.32	2.09	2.71	4.33	1.95	2.48	3.85
.05	1.92	2.47	3.85	1.95	2.45	3.86	1.80	2.25	3.37
		Ave-LM$_1^{\beta\alpha}$			Ave-LM$_2^{\beta\alpha}$			Ave-LM$_3^{\beta\alpha}$	
.25	5.34	6.46	8.95	5.33	6.40	8.85	5.29	6.33	8.82
.15	5.09	6.02	8.34	5.07	6.08	8.22	5.00	5.97	8.02
.05	4.86	5.69	7.64	4.81	5.70	7.56	4.78	5.59	7.40
					$p = 3, r = 1$				
		Ave-LM$_1^{\beta}$			Ave-LM$_2^{\beta}$			Ave-LM$_3^{\beta}$	
.25	3.92	4.88	7.22	3.82	4.78	7.26	3.73	4.65	6.91
.15	3.69	4.59	6.57	3.64	4.48	6.46	3.51	4.32	6.26
.05	3.47	4.25	5.87	3.45	4.16	5.79	3.31	4.01	5.78
		Ave-LM$_1^{\beta\alpha}$			Ave-LM$_2^{\beta\alpha}$			Ave-LM$_3^{\beta\alpha}$	
.25	8.12	9.47	12.52	8.05	9.35	12.41	7.97	9.33	12.07
.15	7.79	8.89	11.46	7.71	8.83	11.44	7.68	8.74	11.30
.05	7.42	8.40	10.55	7.38	8.30	10.61	7.37	8.27	10.38
					$p = 3, r = 2$				
		Ave-LM$_1^{\beta}$			Ave-LM$_2^{\beta}$			Ave-LM$_3^{\beta}$	
.25	3.82	4.78	7.12	3.76	4.72	6.97	3.65	4.50	6.54
.15	3.59	4.48	6.46	3.57	4.36	6.19	3.42	4.15	5.81
.05	3.41	4.14	5.78	3.37	4.03	5.70	3.23	3.80	5.19
		Ave-LM$_1^{\beta\alpha}$			Ave-LM$_2^{\beta\alpha}$			Ave-LM$_3^{\beta\alpha}$	
.25	11.97	13.63	16.92	11.90	13.52	16.62	11.85	13.37	16.87
.15	11.54	13.01	15.98	11.50	12.93	15.73	11.40	12.77	15.77
.05	11.13	12.36	14.89	11.08	12.32	14.74	10.98	12.22	14.64
					$p = 4, r = 1$				
		Ave-LM$_1^{\beta}$			Ave-LM$_2^{\beta}$			Ave-LM$_3^{\beta}$	
.25	5.38	6.46	9.16	5.32	6.47	8.83	5.22	6.24	8.53
.15	5.10	6.05	8.43	5.09	6.02	8.05	4.98	5.89	7.95
.05	4.84	5.65	7.78	4.85	5.63	7.35	4.75	5.48	7.23
		Ave-LM$_1^{\beta\alpha}$			Ave-LM$_2^{\beta\alpha}$			Ave-LM$_3^{\beta\alpha}$	
.25	10.59	12.17	15.48	10.66	12.09	15.37	10.52	12.02	15.51
.15	10.21	11.66	14.43	10.22	11.56	14.41	10.16	11.44	14.33
.05	9.88	11.02	13.42	9.79	10.94	13.31	9.72	10.91	13.30

(continued)

Table 1. *(continued)*

					$p = 4, r = 2$				
		Ave-LM_1^{β}			Ave-LM_2^{β}			Ave-LM_3^{β}	
.25	6.63	7.90	10.53	6.71	7.86	10.54	6.52	7.71	10.40
.15	6.35	7.45	9.92	6.37	7.37	9.76	6.24	7.17	9.43
.05	6.05	6.96	9.12	6.09	6.93	9.12	5.92	6.72	8.68
		Ave-$LM_1^{\beta\alpha}$			Ave-$LM_2^{\beta\alpha}$			Ave-$LM_3^{\beta\alpha}$	
.25	16.76	18.64	22.46	16.70	18.45	22.45	16.71	18.47	22.02
.15	16.23	17.90	21.32	16.18	17.75	21.11	16.19	17.68	20.72
.05	15.69	17.15	20.03	15.59	17.05	19.81	15.70	16.90	19.72
					$p = 4, r = 3$				
		Ave-LM_1^{β}			Ave-LM_2^{β}			Ave-LM_3^{β}	
.25	5.25	6.27	8.68	5.28	6.38	8.63	5.02	6.00	7.91
.15	4.99	5.89	8.04	4.97	6.00	7.93	4.74	5.55	7.15
.05	4.73	5.52	7.26	4.74	5.59	7.24	4.47	5.17	6.51
		Ave-$LM_1^{\beta\alpha}$			Ave-$LM_2^{\beta\alpha}$			Ave-$LM_3^{\beta\alpha}$	
.25	20.43	22.16	26.34	20.35	22.12	26.25	20.23	22.09	25.93
.15	19.77	21.42	24.94	19.67	21.41	24.91	19.58	21.25	24.74
.05	19.15	20.62	23.60	19.04	20.58	23.68	19.02	20.47	23.58
					$p = 5, r = 1$				
		Ave-LM_1^{β}			Ave-LM_2^{β}			Ave-LM_3^{β}	
.25	6.70	8.00	10.64	6.71	7.98	10.54	6.70	7.91	10.68
.15	6.49	7.52	9.88	6.42	7.50	9.71	6.39	7.47	9.98
.05	6.16	7.08	9.11	6.13	7.03	9.00	6.07	7.01	9.12
		Ave-$LM_1^{\beta\alpha}$			Ave-$LM_2^{\beta\alpha}$			Ave-$LM_3^{\beta\alpha}$	
.25	13.23	14.96	18.50	13.24	14.96	18.26	13.26	14.87	18.29
.15	12.73	14.31	17.42	12.77	14.22	17.22	12.73	14.27	17.18
.05	12.26	13.55	16.36	12.28	13.52	16.18	12.22	13.60	16.04
					$p = 5, r = 2$				
		Ave-LM_1^{β}			Ave-LM_2^{β}			Ave-LM_3^{β}	
.25	9.30	10.79	13.55	9.28	10.74	13.69	9.30	10.76	13.93
.15	8.99	10.25	12.76	8.88	10.20	12.77	8.91	10.18	12.82
.05	8.61	9.70	11.87	8.50	9.59	11.97	8.56	9.61	11.89
		Ave-$LM_1^{\beta\alpha}$			Ave-$LM_2^{\beta\alpha}$			Ave-$LM_3^{\beta\alpha}$	
.25	21.51	23.52	27.67	21.51	23.56	27.69	21.54	23.44	27.48
.15	20.86	22.74	26.39	20.84	22.72	26.34	20.85	22.57	26.23
.05	20.22	21.81	25.07	20.22	21.82	24.97	20.26	21.67	24.91

(continued)

TABLE 1. (*continued*)

	$p = 5, r = 3$								
		Ave-LM_1^β			Ave-LM_2^β			Ave-LM_3^β	
.25	9.34	10.66	13.53	9.33	10.64	13.29	9.26	10.52	13.40
.15	8.98	10.16	12.60	8.95	10.16	12.52	8.85	9.98	12.60
.05	8.60	9.67	11.80	8.54	9.58	11.70	8.47	9.45	11.78
		Ave-$\mathrm{LM}_1^{\beta\alpha}$			Ave-$\mathrm{LM}_2^{\beta\alpha}$			Ave-$\mathrm{LM}_3^{\beta\alpha}$	
.25	27.31	29.46	34.13	27.27	29.49	33.91	27.23	29.56	34.17
.15	26.64	28.60	32.77	26.64	28.61	32.60	26.60	28.64	32.61
.05	25.97	27.66	31.27	25.91	27.65	31.31	25.89	27.70	31.03
	$p = 5, r = 4$								
		Ave-LM_1^β			Ave-LM_2^β			Ave-LM_3^β	
.25	6.56	7.66	10.34	6.69	7.84	10.45	6.39	7.41	9.62
.15	6.22	7.23	9.44	6.37	7.34	9.52	6.05	6.90	8.81
.05	5.97	6.87	8.81	6.07	6.93	8.85	5.76	6.51	8.15
		Ave-$\mathrm{LM}_1^{\beta\alpha}$			Ave-$\mathrm{LM}_2^{\beta\alpha}$			Ave-$\mathrm{LM}_3^{\beta\alpha}$	
.25	30.57	32.82	37.28	30.55	32.93	37.59	30.43	32.82	37.33
.15	29.87	31.96	35.92	29.92	32.07	36.19	29.75	31.80	35.97
.05	29.15	30.95	34.61	29.21	31.00	34.64	29.05	30.84	34.52

5.1. Power

Suppose we have the following local alternative hypothesis:

$\mathcal{H}_n^\beta$: $\delta_n = \bar{d}/n$,

where $\delta_n = \delta$ in equation (1).

The asymptotic power function is driven by the Lagrange multiplier under the local alternative hypothesis. This can be defined alternatively as follows:

$$\begin{aligned} g_n^\beta(\tilde{\theta},\tau) &= n^{-1} \sum_{t=[n\tau]+1}^{n} R_{12t}(1-\tau)\tilde{v}_t' \\ &= n^{-1}\Bigg(\sum_{t=[n\tau]+1}^{n} x_{2t-1}v_t' \\ &\qquad - \sum_{t=[n\tau]+1}^{n} x_{2t-1}x_{2t-1}'\left(\sum_{t=1}^{n} x_{2t-1}x_{2t-1}\right)^{-1}\sum_{t=1}^{n} x_{2t-1}v_t'\Bigg) + o_p(1) \end{aligned}$$

uniformly in $\tau \in \tau^*$, where $R_{12t}(1-\tau) = x_{2t-1} - \sum_{t=[n\tau]+1}^{n} x_{2t-1}z_{t-1}'(\sum_{t=[n\tau]+1}^{n} z_{t-1}z_{t-1}')^{-1}z_{t-1}$.

TABLE 2. Asymptotic critical values for Exp-LM^{β}, Exp-$LM^{\beta\alpha}$

$\underline{\tau}$	10%	5%	1%	10%	5%	1%	10%	5%	1%
					$p = 2, r = 1$				
		Exp-LM_1^{β}			Exp-LM_2^{β}			Exp-LM_3^{β}	
.25	1.48	2.01	3.35	1.50	2.02	3.32	1.49	1.99	3.27
.15	1.49	2.02	3.33	1.50	2.02	3.33	1.50	2.01	3.28
.05	1.51	2.00	3.22	1.52	2.05	3.32	1.53	2.02	3.22
		Exp-$LM_1^{\beta\alpha}$			Exp-$LM_2^{\beta\alpha}$			Exp-$LM_3^{\beta\alpha}$	
.25	3.45	4.19	5.79	3.43	4.18	5.80	3.47	4.21	5.87
.15	3.54	4.26	5.87	3.51	4.25	5.85	3.55	4.26	5.88
.05	3.60	4.32	6.03	3.60	4.34	5.96	3.64	4.31	6.01
					$p = 3, r = 1$				
		Exp-LM_1^{β}			Exp-LM_2^{β}			Exp-LM_3^{β}	
.25	2.54	3.26	4.71	2.54	3.17	4.77	2.52	3.19	4.71
.15	2.59	3.30	4.74	2.60	3.25	4.79	2.56	3.24	4.81
.05	2.63	3.29	4.70	2.64	3.32	4.72	2.61	3.28	4.79
		Exp-$LM_1^{\beta\alpha}$			Exp-$LM_2^{\beta\alpha}$			Exp-$LM_3^{\beta\alpha}$	
.25	5.13	6.02	8.16	5.06	5.99	8.03	5.10	6.03	7.89
.15	5.26	6.15	8.19	5.21	6.13	8.11	5.25	6.13	7.98
.05	5.41	6.27	8.34	5.41	6.30	8.21	5.42	6.27	8.16
					$p = 3, r = 2$				
		Exp-LM_1^{β}			Exp-LM_2^{β}			Exp-LM_3^{β}	
.25	2.54	3.20	4.76	2.50	3.19	4.76	2.57	3.28	4.73
.15	2.59	3.21	4.88	2.55	3.19	4.78	2.63	3.31	4.75
.05	2.64	3.26	4.87	2.60	3.24	4.76	2.68	3.32	4.67
		Exp-$LM_1^{\beta\alpha}$			Exp-$LM_2^{\beta\alpha}$			Exp-$LM_3^{\beta\alpha}$	
.25	7.48	8.51	10.66	7.45	8.49	10.54	7.47	8.53	10.62
.15	7.67	8.64	10.97	7.66	8.63	10.81	7.67	8.73	10.79
.05	7.84	8.84	11.15	7.83	8.78	11.06	7.89	8.86	11.06
					$p = 4, r = 1$				
		Exp-LM_1^{β}			Exp-LM_2^{β}			Exp-LM_3^{β}	
.25	3.47	4.17	6.02	3.43	4.17	5.74	3.39	4.11	5.70
.15	3.53	4.21	5.99	3.52	4.18	5.79	3.48	4.20	5.74
.05	3.60	4.32	5.98	3.62	4.28	5.92	3.58	4.25	5.79
		Exp-$LM_1^{\beta\alpha}$			Exp-$LM_2^{\beta\alpha}$			Exp-$LM_3^{\beta\alpha}$	
.25	6.68	7.71	9.81	6.65	7.70	9.66	6.60	7.58	9.79
.15	6.84	7.79	10.04	6.85	7.81	9.68	6.81	7.87	9.88
.05	7.02	7.98	10.06	7.04	7.99	9.83	6.99	7.91	9.97

(continued)

Table 2. *(continued)*

				$p = 4, r = 2$					
		Exp-LM$_1^\beta$			Exp-LM$_2^\beta$			Exp-LM$_3^\beta$	
.25	4.30	5.14	7.07	4.32	5.14	6.89	4.35	5.15	7.05
.15	4.39	5.23	6.93	4.40	5.25	6.85	4.46	5.15	6.99
.05	4.52	5.29	6.99	4.49	5.30	6.96	4.54	5.23	6.96
		Exp-LM$_1^{\beta\alpha}$			Exp-LM$_2^{\beta\alpha}$			Exp-LM$_3^{\beta\alpha}$	
.25	10.29	11.49	14.16	10.31	11.41	14.04	10.34	11.43	13.89
.15	10.55	11.68	14.20	10.55	11.59	14.11	10.56	11.62	14.12
.05	10.84	11.92	14.42	10.75	11.83	14.26	10.77	11.81	14.28
				$p = 4, r = 3$					
		Exp-LM$_1^\beta$			Exp-LM$_2^\beta$			Exp-LM$_3^\beta$	
.25	3.47	4.23	5.82	3.46	4.27	5.90	3.54	4.26	5.85
.15	3.57	4.26	5.95	3.58	4.28	5.81	3.61	4.33	5.86
.05	3.64	4.36	6.01	3.66	4.33	5.86	3.71	4.42	5.90
		Exp-LM$_1^{\beta\alpha}$			Exp-LM$_2^{\beta\alpha}$			Exp-LM$_3^{\beta\alpha}$	
.25	12.37	13.61	16.34	12.35	13.51	16.43	12.36	13.61	16.49
.15	12.64	13.93	16.30	12.62	13.73	16.63	12.68	13.92	16.61
.05	13.02	14.25	16.65	12.96	14.17	16.91	13.01	14.27	16.91
				$p = 5, r = 1$					
		Exp-LM$_1^\beta$			Exp-LM$_2^\beta$			Exp-LM$_3^\beta$	
.25	4.27	5.08	6.84	4.26	5.10	6.89	4.28	5.12	6.95
.15	4.38	5.14	6.94	4.38	5.18	6.90	4.37	5.20	7.08
.05	4.49	5.24	6.98	4.47	5.26	6.82	4.49	5.32	7.11
		Exp-LM$_1^{\beta\alpha}$			Exp-LM$_2^{\beta\alpha}$			Exp-LM$_3^{\beta\alpha}$	
.25	8.19	9.28	11.52	8.21	9.26	11.48	8.20	9.28	11.65
.15	8.38	9.48	11.51	8.43	9.44	11.59	8.41	9.48	11.70
.05	8.58	9.59	11.70	8.57	9.56	11.74	8.61	9.63	11.82
				$p = 5, r = 2$					
		Exp-LM$_1^\beta$			Exp-LM$_2^\beta$			Exp-LM$_3^\beta$	
.25	5.90	6.82	8.80	5.94	6.82	8.95	5.95	6.93	8.98
.15	6.03	6.96	8.94	6.07	6.93	9.03	6.09	7.02	9.05
.05	6.19	7.05	8.90	6.18	7.01	9.11	6.26	7.15	9.01
		Exp-LM$_1^{\beta\alpha}$			Exp-LM$_2^{\beta\alpha}$			Exp-LM$_3^{\beta\alpha}$	
.25	13.02	14.36	17.26	13.11	14.38	17.23	13.12	14.47	16.99
.15	13.38	14.62	17.42	13.39	14.62	17.50	13.45	14.71	17.22
.05	13.69	14.91	17.72	13.68	14.90	17.65	13.76	15.01	17.52

(continued)

Table 2. (*continued*)

$p = 5, r = 3$									
		Exp-LM$_1^\beta$			Exp-LM$_2^\beta$			Exp-LM$_3^\beta$	
.25	5.97	6.88	8.73	5.94	6.81	8.78	5.95	6.79	8.97
.15	6.13	7.05	8.94	6.07	6.96	8.72	6.09	7.01	9.00
.05	6.26	7.17	9.02	6.18	7.05	8.85	6.25	7.16	9.11
		Exp-LM$_1^{\beta\alpha}$			Exp-LM$_2^{\beta\alpha}$			Exp-LM$_3^{\beta\alpha}$	
.25	16.53	17.81	20.82	16.46	17.87	20.72	16.55	17.90	20.74
.15	16.86	18.17	21.09	16.79	18.20	20.94	16.87	18.20	21.07
.05	17.21	18.57	21.43	17.17	18.52	21.35	17.26	18.57	21.26
$p = 5, r = 4$									
		Exp-LM$_1^\beta$			Exp-LM$_2^\beta$			Exp-LM$_3^\beta$	
.25	4.29	5.14	6.92	4.34	5.19	7.09	4.46	5.21	6.86
.15	4.44	5.23	6.86	4.46	5.27	7.07	4.58	5.29	7.09
.05	4.54	5.32	7.04	4.57	5.35	7.15	4.64	5.37	7.14
		Exp-LM$_1^{\beta\alpha}$			Exp-LM$_2^{\beta\alpha}$			Exp-LM$_3^{\beta\alpha}$	
.25	18.32	19.92	22.59	18.28	19.84	22.71	18.42	19.90	22.74
.15	18.76	20.25	23.00	18.78	20.25	23.08	18.84	20.27	23.00
.05	19.30	20.70	23.55	19.25	20.71	23.83	19.36	20.71	23.63

The following lemma can be verified analogously to Lemma 2. We denote τ_0 as the true break point.

LEMMA 7. *Under* $\mathcal{H}_n^\beta$ *and Assumptions* 1 *and* 2,

$$n^{-1}\sum_{t=[n\tau]+1}^{n} x_{2t-1}v_t' \Rightarrow \int_\tau^1 W_2\, dB_1' + \left(\int_{\max(\tau,\tau_0)}^1 W_2 W_2'\, ds\right)\bar{d}(\alpha'\Sigma^{-1}\alpha)^{1/2}. \tag{32}$$

The power of the test mainly comes from the decision error $\bar{d}$. If τ_0 is known, it removes uncertainty. This case generates the power envelope for the tests that treat τ as unknown.

Figure 1 depicts the asymptotic local power function of the tests for structural change of the cointegrating vector from the following bivariate model:

$$\begin{pmatrix}\Delta x_{1t}\\ \Delta x_{2t}\end{pmatrix} = \begin{pmatrix}-1\\ 0\end{pmatrix}\begin{pmatrix}1\\ -1 + \bar{d}/n\{t \geq [n\tau] + 1\}\end{pmatrix}'\begin{pmatrix}x_{1t-1}\\ x_{2t-1}\end{pmatrix} + \begin{pmatrix}e_{1t}\\ e_{2t}\end{pmatrix},$$

where $\{e_{1t}, e_{2t}\}'$ are i.i.d. and standard normal with covariance ρ.

Monte Carlo experiments are based on a sample size of 250 and 3,000 replications for each $\bar{d} = 0,2,\ldots,30$, and $\rho = 0.2$. The break point τ is assumed to be

TABLE 3. Asymptotic critical values for Sup-LM$^{\beta}$, Sup-LM$^{\beta\alpha}$

$\underline{\tau}$	10%	5%	1%	10%	5%	1%	10%	5%	1%
					$p = 2, r = 1$				
		Sup-LM$_1^{\beta}$			Sup-LM$_2^{\beta}$			Sup-LM$_3^{\beta}$	
.25	6.92	8.38	11.80	6.88	8.34	11.69	7.17	8.63	12.23
.15	7.64	9.10	12.51	7.62	9.09	12.58	8.05	9.46	12.93
.05	8.61	9.95	13.28	8.54	10.04	13.56	9.04	10.52	13.92
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	11.52	13.35	16.99	11.44	13.26	17.09	11.71	13.46	17.35
.15	12.42	14.19	18.07	12.46	14.23	18.29	12.64	14.37	18.37
.05	13.57	15.23	19.47	13.58	15.22	19.67	13.80	15.55	19.65
					$p = 3, r = 1$				
		Sup-LM$_1^{\beta}$			Sup-LM$_2^{\beta}$			Sup-LM$_3^{\beta}$	
.25	9.75	11.60	15.27	9.71	11.46	15.24	9.93	11.67	14.99
.15	10.72	12.50	16.09	10.71	12.52	16.24	10.85	12.55	16.49
.05	11.75	13.30	16.91	11.71	13.45	17.17	11.93	13.54	17.56
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	15.43	17.61	22.38	15.34	17.40	21.93	15.54	17.63	21.60
.15	16.65	18.60	23.33	16.52	18.65	23.29	16.74	18.71	22.82
.05	17.86	19.98	24.47	17.82	19.94	24.30	18.00	20.02	24.01
					$p = 3, r = 2$				
		Sup-LM$_1^{\beta}$			Sup-LM$_2^{\beta}$			Sup-LM$_3^{\beta}$	
.25	9.68	11.41	15.23	9.54	11.22	15.11	10.05	11.80	15.33
.15	10.57	12.18	16.24	10.44	12.03	15.90	10.98	12.78	16.06
.05	11.58	13.29	17.42	11.50	13.27	17.07	12.07	13.67	17.11
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	20.56	22.80	27.41	20.47	22.71	27.28	20.53	22.86	27.56
.15	21.78	23.96	29.09	21.65	23.87	28.64	21.88	24.22	28.72
.05	23.15	25.35	30.70	23.02	25.30	30.28	23.32	25.36	30.32
					$p = 4, r = 1$				
		Sup-LM$_1^{\beta}$			Sup-LM$_2^{\beta}$			Sup-LM$_3^{\beta}$	
.25	12.09	13.93	18.01	12.02	13.75	17.62	12.09	13.77	17.42
.15	12.98	14.93	18.77	13.02	14.78	18.64	13.12	14.94	18.24
.05	14.23	16.08	20.06	14.20	15.84	19.70	14.21	16.04	19.43
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	19.00	21.26	26.14	18.99	21.13	25.91	18.97	21.13	26.01
.15	20.24	22.45	27.38	20.16	22.46	27.12	20.19	22.45	27.06
.05	21.57	23.77	28.42	21.52	23.73	27.83	21.53	23.66	28.23

(continued)

Table 3. (*continued*)

				$p = 4, r = 2$					
		Sup-LM$_1^{\beta}$			Sup-LM$_2^{\beta}$			Sup-LM$_3^{\beta}$	
.25	14.02	15.82	20.26	13.93	15.91	19.97	14.29	16.26	20.42
.15	15.04	17.08	20.90	14.94	16.92	21.07	15.41	17.19	21.53
.05	16.18	18.20	21.71	16.16	17.99	22.14	16.46	18.30	22.22
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	26.64	29.20	35.31	26.66	29.17	34.71	26.80	29.36	34.65
.15	28.03	30.58	36.10	28.05	30.36	35.52	28.12	30.47	35.87
.05	29.60	32.16	37.12	29.44	31.85	37.14	29.57	31.93	37.37
				$p = 4, r = 3$					
		Sup-LM$_1^{\beta}$			Sup-LM$_2^{\beta}$			Sup-LM$_3^{\beta}$	
.25	11.99	13.73	17.54	12.02	13.88	17.62	12.30	14.09	17.83
.15	12.96	14.79	18.50	12.98	14.73	18.44	13.34	15.18	18.72
.05	14.11	15.83	19.74	14.19	15.83	19.36	14.56	16.47	20.20
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	31.01	33.59	39.23	30.85	33.43	39.51	31.06	33.74	39.89
.15	32.31	35.02	40.31	32.29	34.79	40.92	32.56	35.17	40.65
.05	34.20	36.94	41.91	34.11	36.73	42.18	34.41	36.92	42.66
				$p = 5, r = 1$					
		Sup-LM$_1^{\beta}$			Sup-LM$_2^{\beta}$			Sup-LM$_3^{\beta}$	
.25	14.08	16.06	20.32	14.02	16.06	19.84	14.18	16.05	20.43
.15	15.10	17.21	21.40	15.05	16.97	21.13	15.27	17.13	21.45
.05	16.28	18.20	22.13	16.22	18.02	21.98	16.34	18.41	22.11
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	22.35	24.74	29.37	22.28	24.74	29.51	22.34	24.77	30.01
.15	23.59	25.96	30.38	23.53	25.91	30.68	23.52	26.11	31.07
.05	24.93	27.23	31.73	24.98	27.00	31.60	25.00	27.29	32.13
				$p = 5, r = 2$					
		Sup-LM$_1^{\beta}$			Sup-LM$_2^{\beta}$			Sup-LM$_3^{\beta}$	
.25	17.65	19.68	24.25	17.72	19.69	24.76	17.95	20.06	24.89
.15	18.85	20.93	25.37	18.80	20.83	25.63	19.10	21.19	25.76
.05	20.05	22.12	26.33	20.05	22.04	26.54	20.38	22.44	26.69
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	32.63	35.39	41.40	32.66	35.50	41.83	32.81	35.68	41.24
.15	34.12	36.87	42.79	34.14	36.80	43.06	34.28	37.16	42.57
.05	35.77	38.42	44.18	35.72	38.35	44.08	35.95	38.51	44.09

(*continued*)

Table 3. (*continued*)

				$p = 5, r = 3$					
		Sup-LM$_1^\beta$			Sup-LM$_2^\beta$			Sup-LM$_3^\beta$	
.25	17.76	19.91	24.30	17.56	19.65	24.09	17.82	19.92	24.69
.15	18.97	21.14	25.53	18.77	20.78	24.90	18.96	21.13	25.94
.05	20.18	22.29	26.47	19.99	22.08	26.16	20.43	22.53	26.71
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	39.75	42.68	48.95	39.66	42.62	48.91	39.88	42.72	48.68
.15	41.33	44.20	50.20	41.22	44.20	49.94	41.40	44.19	50.23
.05	42.97	45.91	51.91	43.03	45.72	51.49	43.12	45.87	51.52
				$p = 5, r = 4$					
		Sup-LM$_1^\beta$			Sup-LM$_2^\beta$			Sup-LM$_3^\beta$	
.25	13.82	15.73	19.94	13.91	15.86	19.97	14.52	16.31	20.23
.15	14.93	16.81	20.74	15.00	16.92	21.11	15.58	17.37	21.30
.05	16.17	18.04	22.10	16.21	18.18	21.88	16.62	18.62	22.72
		Sup-LM$_1^{\beta\alpha}$			Sup-LM$_2^{\beta\alpha}$			Sup-LM$_3^{\beta\alpha}$	
.25	43.41	46.70	52.45	43.48	46.58	52.69	43.56	46.82	52.70
.15	45.13	48.14	54.30	45.19	48.17	54.52	45.44	48.48	54.24
.05	47.16	50.14	56.33	47.04	50.33	56.96	47.59	50.40	56.41

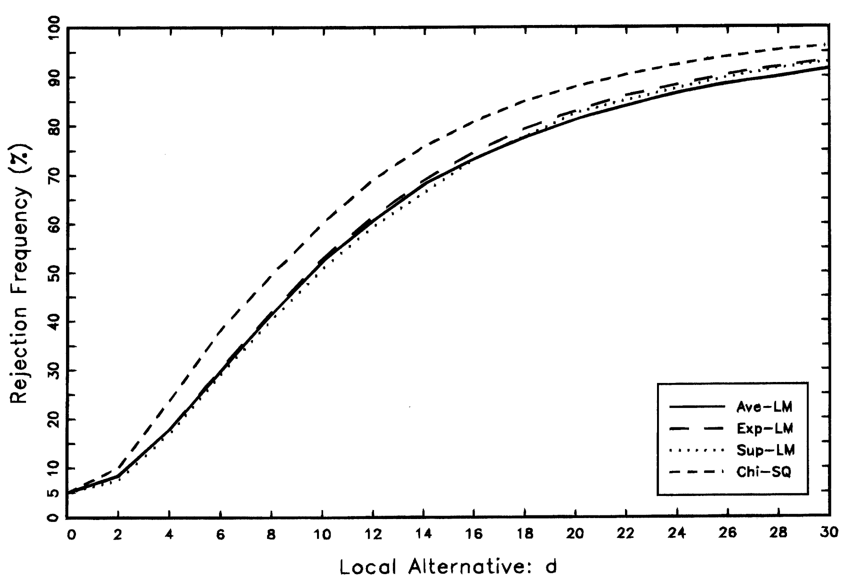

Figure 1. Power function ($N = 250$, 5% size).

uniformly distributed over [.15,.85]. The chi-square test, which is based on the true break point, rejects 88.0% of the null hypothesis at the 5% size when the local alternative $\bar{d}$ is 20. Ave-LM, Exp-LM, and Sup-LM tests reject 81.3%, 83.1%, and 82.6%, respectively.

5.2. Size Distortion

Suppose we have the following local alternative hypothesis:

$\mathcal{H}_n^{\alpha}$: $\epsilon_n = \bar{e}/\sqrt{n}$,

where $\epsilon_n = \epsilon$ in equation (12).

Figure 2 shows the size distortion of the tests for structural change of the cointegrating vector from the following bivariate model:

$$\begin{pmatrix} \Delta x_{1t} \\ \Delta x_{2t} \end{pmatrix} = \begin{pmatrix} -1 + \bar{e}/\sqrt{n}\{t \geq [n\tau] + 1\} \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}' \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \end{pmatrix} + \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}.$$

Monte Carlo experiments are based on the same parameters as in Section 5.1 except $\bar{e} = 0,1,2,\ldots,10$. Also, τ is uniformly distributed over [.15,.85]. The chi-square test at the 5% size rejects 5.8% of the null hypothesis at $\bar{e} = 5$. Ave-LM, Exp-LM, and Sup-LM tests reject 5.8%, 6.1%, and 6.2%, respectively. Hence,

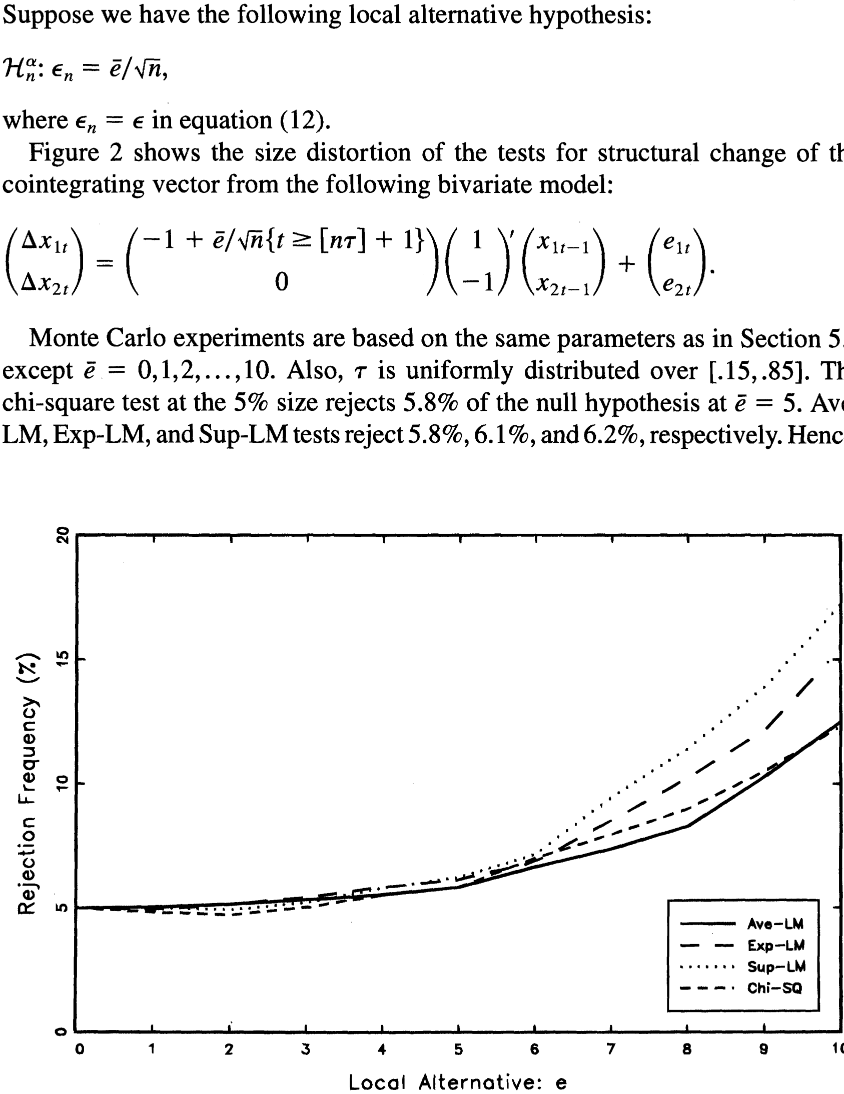

FIGURE 2. Size distortion ($N = 250$, 5% size).

small sample experiments show that the Exp-LM test has more power than other optimal tests whereas the Ave-LM test has smaller size distortion.

6. EMPIRICAL APPLICATION

In this section, our stability tests are applied to the U.S. money demand equation for the period 1900–1985. Specifically, we test structural change of the income elasticity and the interest semielasticity of money demand. This question was posed by Lucas (1988). Although the instability of the postwar U.S. money demand equation has been raised by many authors, Lucas was interested in a stable equilibrium relationship over a century.

We use the data set that was constructed by Lucas (1988): m is M1, p the price index, y real income, and r the short-run interest rate. All variables are in logarithms except the short-run interest rate. We assume that the change point is known to lie in [0.15, 0.85]. All empirical work is done in Gauss, and it is replicable with any software that is capable of matrix operations.[1]

Because the real money balances and real income contain growth terms, the model of trend in the DGP is used. The short-run interest rate is regarded as integrated because the augmented Dickey–Fuller test cannot reject the unit root hypothesis. Johansen's cointegration test rejects the null hypothesis of no cointegration at each VAR lag length from 1 to 9, which suggests a long-run cointegrating relationship of the money demand equation.

Johansen's test is also applied to real income and the interest rate, but the null hypothesis of no cointegration is maintained at each VAR lag length. This result supports the normalization condition that is made with respect to the real money balances. When the lag length picked is 3 by Akaike information criterion, the long-run relationship and adjustment coefficients are estimated as follows:

$$\begin{pmatrix} \Delta(m-p)_t \\ \Delta y_t \\ \Delta r_t \end{pmatrix} = \begin{pmatrix} -.24_{(.08)} \\ -.07_{(.16)} \\ -3.81_{(.12)} \end{pmatrix} \begin{pmatrix} 1 \\ -0.99_{(.03)} \\ .11_{(.01)} \end{pmatrix}' \begin{pmatrix} (m-p)_{t-1} \\ y_{t-1} \\ r_{t-1} \end{pmatrix} + \cdots,$$

$$LR(\mathcal{H}_0\colon rank(\Pi) = 0) = 32.65^*,$$

$$LR(\mathcal{H}_0\colon rank(\Pi) = 1) = 4.58,$$

$$LR(\mathcal{H}_0\colon rank(\Pi) = 2) = 1.28,$$

$$\text{Ave-LM}_n^\beta = 1.59, \qquad \text{Exp-LM}_n^\beta = 1.20, \qquad \text{Sup-LM}_n^\beta = 6.87,$$

$$\text{Ave-LM}_n^\alpha = 6.16^*, \qquad \text{Exp-LM}_n^\alpha = 5.16^*, \qquad \text{Sup-LM}_n^\alpha = 13.06,$$

$$\text{Ave-LM}_n^{\beta\alpha} = 7.76, \qquad \text{Exp-LM}_n^{\beta\alpha} = 5.50, \qquad \text{Sup-LM}_n^{\beta\alpha} = 13.82,$$

where standard errors are in parentheses and * denotes 5% significant.

Our tests cannot reject the stability of the cointegrating vector β at the 5% size. In Figure 3a, LM statistics show some spikes in the 1930's, but they are not

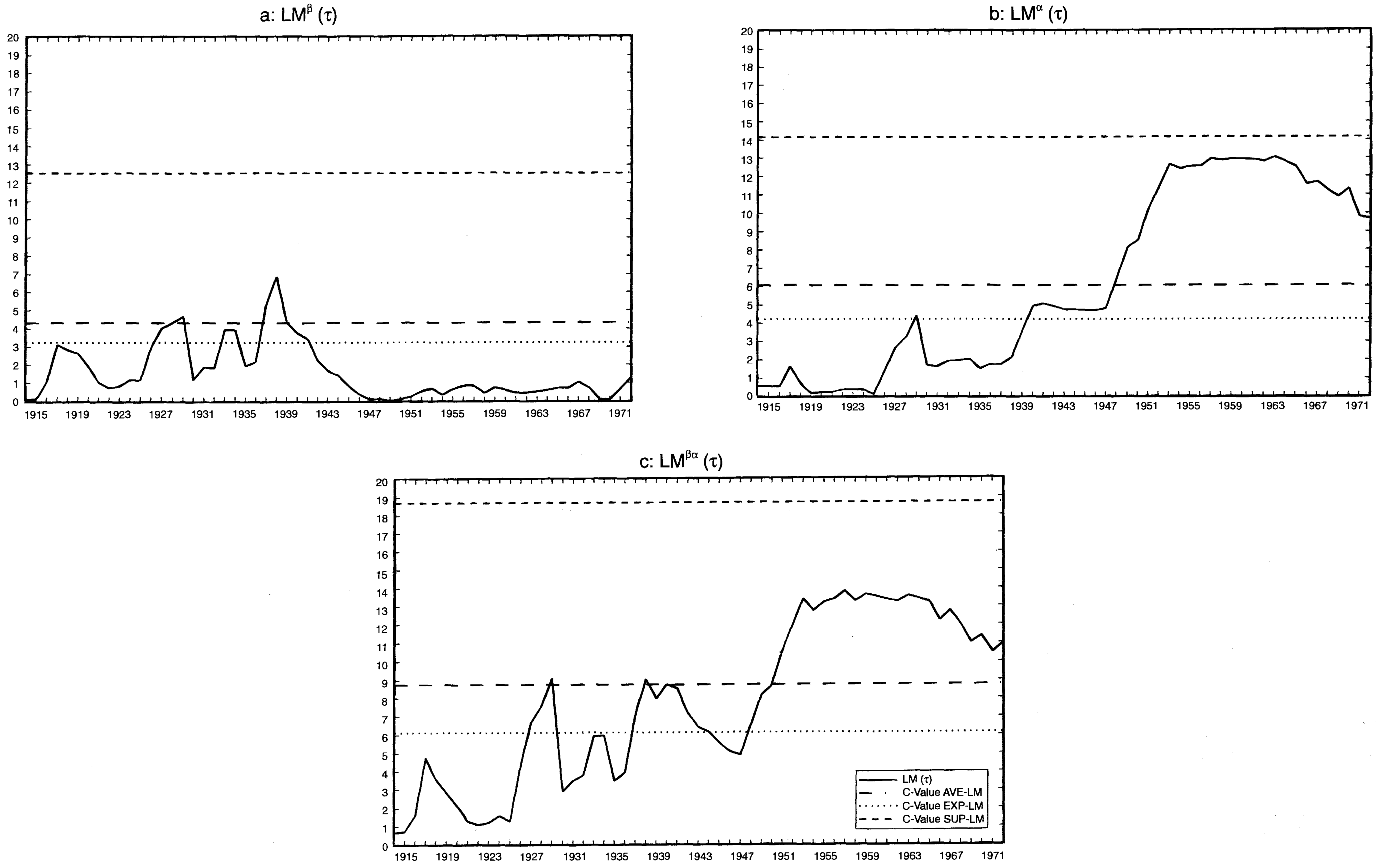

FIGURE 3. Stability tests of the money demand equation: 1900–1985.

significant. Hence, the U.S. money demand equation has a stable long-run relationship for the period 1900–1985. This result corresponds to the conclusions of Lucas (1988) and Stock and Watson(1993).

On the other hand, Ave-LM and Exp-LM tests reject the stability of the adjustment vectors α at the 5% size whereas the Sup-LM test does not. Figure 3b shows that LM statistics are close to the 5% critical value of the Sup-LM test after the 1950's. Lucas (1988) indicates that the real money balances do not grow at the same rate as real income after the 1950's, which may affect the stability of α because the change in the growth rate lowers the speed of adjustment to the equilibrium.

7. CONCLUDING REMARKS

This paper has proposed new tests for structural change of the cointegrating vector and the adjustment vector in the ECM with an unknown change point. Tests for structural change of the cointegrating vector have nonstandard asymptotic distributions that are different from those found by Andrews and Ploberger (1994). In contrast, tests for structural change of the adjustment vector have the same distributions that have been found for models with stationary variables. We have also shown that detrending methods change the distribution theory of the tests for structural change of the cointegrating vector.

The crucial condition made in this paper is the normalization of the cointegration space. Without this condition, the cointegrating vector cannot be identified even though the cointegration space can be identified. It is important to identify the cointegrating vector if we wish to test its stability. The normalization condition is often useful in empirical studies and has been used by many authors.

NOTE

1. A Gauss program can be requested from the author.

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APPENDIX A: MATHEMATICAL PROOFS

In the Appendix, we denote $\|A\|_q = (E|A^q|)^{1/q}$, $|A| = (\operatorname{tr} A'A)^{1/2}$, $\sup_{\tau\in T^*} o_{p\tau}(1) = o_p(1)$, and $\sup_{\tau\in T^*} O_{p\tau}(1) = O_p(1)$. We also denote $\check{z}_t = (w_t', z_t')'$, where $z_t = (\Delta x_t', \dots, \Delta x_{t-l+2}')'$. We use the following lemmas. (For the proof of Lemma 8, see Hall and Heyde, 1980, p. 143; for the proof of Lemma 9, see Andrews, 1988, p. 460, and Andrews, 1993, pp. 846–847).

LEMMA 8. (*Hall and Heyde*, 1980). *If* $\{z_t\}$ *is uniformly square integrable, then*

$$\sup_{t\le n} n^{-1/2}|z_t| \to^p 0.$$

LEMMA 9. (*Andrews*, 1988, 1993). *If* $\{v_t\}$ *is a uniformly integrable* L^1*-mixingale, then*

$$E\sup_{s\le n}\left|n^{-1}\sum_{t=1}^{s} v_t\right| \to 0, \quad and \quad \sup_{s\le n} n^{-1}\left|\sum_{t=1}^{s} v_t\right| \to^p 0.$$

Proof of Lemma 1. By the invariance principle of Phillips and Durlauf (1986).

$$n^{-1/2}\sum_{t=1}^{[ns]} u_t \Rightarrow W(s) = BM(\Sigma).$$

$$n^{-1/2} x_{[ns]} \Rightarrow C(1)W(s). \tag{A.1}$$

We need to show

$$P\left(\sup_{s\in[0,1]} n^{-1/2}\left|x_{[ns]} - C(1)\sum_{t=1}^{[ns]} u_t\right| > \epsilon\right) \le P\left(\sup_{s\in[0,1]} n^{-1/2}|\Phi(L)u_{[ns]}| > \epsilon\right) \to 0.$$

If $\{\Phi(L)u_t\}$ is uniformly square integrable, we can apply Lemma 8.

$$\|\Phi(L)u_t\|_2 \le \sum_{j=0}^{\infty}|\Phi_j|\|u_t\|_2 \le \sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}|C_k|\|u_t\|_2 = \sum_{k=1}^{\infty} k|C_k|(E|u_0|^2)^{1/2} < \infty.$$

In the same way, we can show $\|\Delta x_t\|_2 < \infty$ and $\|w_t\|_2 < \infty$ if $\sum_{k=1} k|C_k| < \infty$, $\sup_{\theta\in\Theta}|\theta| < \infty$, and $E|u_0|^2 < \infty$. Thus, $\sup_{t\le n} n^{-1/2}|\check{z}_t| \to^p 0$.

$$n^{-1/2}\sum_{t=1}^{[ns]} w_t \Rightarrow \gamma'\Phi(1)W(s). \tag{A.2}$$

We need to show

$$P\left(\sup_{s\in[0,1]} n^{-1/2}\left|\sum_{t=1}^{[ns]} w_t - \gamma'\Phi(1)\sum_{t=1}^{[ns]} u_t\right| > \epsilon\right)$$
$$\le P\left(\sup_{s\in[0,1]} n^{-1/2}|\gamma'\Phi_1(L)u_{[ns]}| > \epsilon\right) \to 0,$$

where $\Phi_1(L) = (\Phi(L) - \Phi(1))/(1-L)$. Here $\sum_{k=1}^{\infty} k^2|C_k| < \infty$, $\sup_{\theta\in\Theta}|\theta| < \infty$, and $E|u_0|^2 < \infty$ imply

$$\|\gamma'\Phi_1(L)u_t\|_2 \le \sup_\gamma|\gamma|\left\|\sum_{j=0}^{\infty}\sum_{k=j+2}^{\infty}(k-j-1)C_k u_t\right\|_2 \le \sup_\gamma|\gamma|\sum_{k=1}^{\infty}k^2|C_k|\|u_0\|_2 < \infty.$$

$$n^{-1/2}R_{12t}(\tau) \Rightarrow C_2(1)W(s) \equiv W_2(s). \tag{A.3}$$

We need to show $\sup_{s\in[0,1],\tau\in\mathcal{T}^*}|n^{-1/2}\sum_{t=1}^{[n\tau]}x_t z_t'(\sum_{t=1}^{[n\tau]}z_t z_t')^{-1}z_{[ns]}| \to^p 0$.

If we can show that $v_t = \{z_t z_t' - E(z_0 z_0')\}$ is a uniformly integrable L^1-mixingale, we can apply Lemma 9 to show $\sup_{\tau\in\mathcal{T}^*}n^{-1}|\sum_{t=1}^{[n\tau]}v_t| \to^p 0$. We denote $v_{1t} = \{\Delta x_t\Delta x_t' - E(\Delta x_0\Delta x_0')\}$, $v_{2t} = \{w_t w_t' - E(w_0 w_0')\}$, and $v_{3t} = \{\Delta x_t w_t' - E(\Delta x_0 w_0')\}$.

$$\begin{aligned}\|E(v_{1t}|F_{t-m})\|_1 &= \left\|\sum_{j=m}^{\infty}C_j(u_{t-j}u_{t-j}' - \Sigma)C_j' + \sum_{k\neq j}\sum_{j,k=m}^{\infty}C_j u_{t-j}u_{t-k}'C_k'\right\|_1\\ &\le \sum_{j=m}^{\infty}|C_j|^2(\|u_0\|_2^2 + |\Sigma|) + \sum_{k\neq j}\sum_{j,k=m}^{\infty}|C_j||C_k|\|u_0\|_2^2\\ &\le 2\|u_0\|_2^2\left(\sum_{k=m}^{\infty}|C_k|\right)^2 \to 0 \quad \text{as } m\to\infty\end{aligned}$$

because $\sum_{k=0}^{\infty}|C_k| < \infty$. Thus, v_t is an L^1-mixingale.

We can also show that $\|E(v_{2t}|F_{t-m})\|_1 \to 0$ and $\|E(v_{3t}|F_{t-m})\|_1 \to 0$ as $m\to\infty$ if $\sum_{k=1}^{\infty}k|C_k| < \infty$ and $\sup_{\theta\in\Theta}|\theta| < \infty$. Because $\|\check{z}_t\|_2 < \infty$, $(v_{1t}', v_{2t}', v_{3t}')'$ is uniformly integrable. From Lemma 9, we have $\sup_{\tau\in\mathcal{T}^*}n^{-1}|\sum_{t=1}^{[n\tau]}(\check{z}_t\check{z}_t' - E(\check{z}_0\check{z}_0'))| \to^p 0$.

$$\sup_{\tau\in\mathcal{T}^*}n^{-1}\sum_{t=1}^{[n\tau]}x_t z_t' = O_p(1). \tag{A.4}$$

$$n^{-1}\sum_{t=1}^{[n\tau]}x_t w_t' = n^{-1}\sum_{t=1}^{[n\tau]}(x_{t-1} + \Delta x_t)w_t' \Rightarrow \int_0^\tau C(1)WdW'\Phi(1)'\gamma + \tau\Lambda_1,$$

where $\Lambda_1 = E(\Delta x_0 w_0') + E(\eta_0 w_1')$, and $\eta_t = \Phi(L)u_t$.

$$\begin{aligned}n^{-1}\sum_{t=1}^{[n\tau]}x_t\Delta x_{t-i}' &= n^{-1}\sum_{t=1}^{[n\tau]}(x_{t-i-1} + \Delta x_{t-i} + \cdots + \Delta x_t)\Delta x_{t-i}'\\ &\Rightarrow \int_0^\tau C(1)WdW'C(1)' + \tau\Lambda_{2i},\end{aligned}$$

where $\Lambda_{2i} = E(\Delta x_0\Delta x_0' + \cdots + \Delta x_i\Delta x_0') + E(\eta_0\Delta x_1')$ for $i = 0,1,\ldots,l-2$.

Because $\sup_{s\in[0,1]}n^{-1/2}|z_{[ns]}| \to^p 0$, we have

$$\sup_{s,\tau}\left|n^{-1}\sum_{t=1}^{[n\tau]}x_t z_t'\left(n^{-1}\sum_{t=1}^{[n\tau]}z_t z_t'\right)^{-1}n^{-1/2}z_{[ns]}\right| \to^p 0. \qquad \blacksquare$$

Proof of Lemma 2. Because we have $n^{-1/2}R_{12[ns]}(\tau) \Rightarrow W_2(s)$, and $\{u_t\}$ is a martingale difference sequence, we can apply the weak convergence theorem by Hansen (1992b) for the proof of (21). (22) holds by the continuous mapping theorem (CMT).

By Lemma 9,

$$n^{-1}\sum_{t=1}^{[n\tau]} R_{2t}(\tau)R_{2t}'(\tau) = n^{-1}\sum_{t=1}^{[n\tau]} w_{t-1}w_{t-1}' - n^{-1}\sum_{t=1}^{[n\tau]} w_{t-1}z_{t-1}' \left(n^{-1}\sum_{t=1}^{[n\tau]} z_{t-1}z_{t-1}'\right)^{-1} n^{-1}\sum_{t=1}^{[n\tau]} z_{t-1}w_{t-1}'$$

$$\to^p \tau E(w_0w_0') - \tau E(w_0z_0')E(z_0z_0')^{-1}E(z_0w_0') \equiv \tau Q$$

uniformly in $\tau \in \tau^*$. ■

LEMMA 10. (*Johansen*, 1988).

$$n(\tilde{\beta} - \beta) \Rightarrow \left(\int_0^1 W_2W_2'\right)^{-1} \int_0^1 W_2\,dW_1'(\alpha'\Sigma^{-1}\alpha)^{-1},$$

$$\sqrt{n}\,\mathrm{vec}(\tilde{\alpha} - \alpha)' \Rightarrow (\Sigma^{1/2} \otimes Q^{-1})K(1),$$

where

$$\begin{pmatrix} n^{-1/2}\alpha'\Sigma^{-1}\sum_{t=1}^{[ns]} u_t \\ n^{-1/2}C_2(1)\sum_{t=1}^{[ns]} u_t \\ n^{-1/2}\sum_{t=1}^{[ns]} (\Sigma^{-1/2'}u_t \otimes R_{2t}(1)) \end{pmatrix} \Rightarrow \begin{pmatrix} W_1(s) \\ W_2(s) \\ K(s) \end{pmatrix}$$

$$= BM\begin{pmatrix} \alpha'\Sigma^{-1}\alpha & 0 & 0 \\ 0 & C_2(1)\Sigma C_2'(1) & 0 \\ 0 & 0 & I\otimes Q \end{pmatrix},$$

where $R_{2t}(1) = w_{t-1} - \sum_{t=1}^n w_{t-1}z_{t-1}(\sum_{t=1}^n z_{t-1}z_{t-1}')^{-1}z_{t-1}$.

Proof of Lemma 10. The distribution theory of $\tilde{\beta}$ has been found by Johansen (1988) in the ECM with a general normalization condition. Hence, Lemma 10 is a special case of Johansen (1988).

Because the restricted MLE satisfies equations (7)–(9),

$$\left(\tilde{\alpha}'\tilde{\Sigma}^{-1} \otimes n^{-1}\sum_{t=1}^n R_{12t}(1)\tilde{u}_t'\right)\mathrm{vec}(I) = 0,$$

and

$$\left(\tilde{\Sigma}^{-1} \otimes n^{-1/2}\sum_{t=1}^n \tilde{R}_{2t}(1)\tilde{u}_t'\right)\mathrm{vec}(I) = 0.$$

From the Taylor series expansion,

$$\begin{pmatrix} n\,\mathrm{vec}(\tilde{\beta} - \beta) \\ \sqrt{n}\,\mathrm{vec}(\tilde{\alpha} - \alpha)' \end{pmatrix} = (-H_n(\theta^*))^{-1} \begin{pmatrix} \left(\alpha'\Sigma^{-1} \otimes n^{-1}\sum_{t=1}^n R_{12t}(1)u_t'\right)\mathrm{vec}(I) \\ \left(\Sigma^{-1} \otimes n^{-1/2}\sum_{t=1}^n R_{2t}(1)u_t'\right)\mathrm{vec}(I) \end{pmatrix},$$

where $\theta^* \in [\theta, \tilde{\theta}]$,

$$-H_n(\theta) = \begin{pmatrix} \alpha'\Sigma^{-1}\alpha \otimes n^{-1}S_{11n}(1) & -H_{12n}(\theta) \\ -H'_{12n}(\theta) & \Sigma^{-1} \otimes S_{22n}(1) \end{pmatrix},$$

and $-H_{12n}(\theta) = n^{-1/2}(\alpha'\Sigma^{-1} \otimes S_{12n}(1)) - n^{-1}(I \otimes n^{-1/2}\sum_{t=1}^{n} R_{2t}(1)u_t'\Sigma^{-1})A$, where A is a rotation of the identity matrix satisfying $A' = \partial\,\mathrm{vec}(\alpha)/\partial\,\mathrm{vec}(\alpha')$.

We will show $S_{12n}(1) = O_p(1)$ in (A.5). Because $\theta^* \to^p \theta$ and $H_{12n}(\theta) \to^p 0$,

$$\begin{pmatrix} n\,\mathrm{vec}(\tilde{\beta} - \beta) \\ \sqrt{n}\,\mathrm{vec}(\tilde{\alpha} - \alpha)' \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha'\Sigma^{-1}\alpha \otimes n^{-1}S_{11n}(1))^{-1}\left(\alpha'\Sigma^{-1} \otimes n^{-1}\sum_{t=1}^{n} R_{12t}(1)u_t'\right)\mathrm{vec}(I) + o_p(1) \\ (\Sigma^{-1} \otimes S_{22n}(1))^{-1}\left(\Sigma^{-1} \otimes n^{-1/2}\sum_{t=1}^{n} R_{2t}(1)u_t'\right)\mathrm{vec}(I) + o_p(1) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \left((\alpha'\Sigma^{-1}\alpha)^{-1}\alpha'\Sigma^{-1} \otimes \left(\int_0^1 W_2W_2'\right)^{-1}\int_0^1 W_2\,dW'\right)\mathrm{vec}(I) \\ (\Sigma^{1/2} \otimes Q^{-1})K(1) \end{pmatrix}.$$

We will show $n^{-1/2}\sum_{t=1}^{[ns]}(\Sigma^{-1/2\prime}u_t \otimes R_{2t}(1)) \Rightarrow K(s)$ for the general case in (A.7). Next, we show that $E(W_1(s)K(s)') = 0$.

$$\lim_{n\to\infty} E\left(n^{-1}\alpha'\Sigma^{-1}\sum_{t=1}^{n} u_t \sum_{t=1}^{n}(u_t'\Sigma^{-1/2} \otimes R_{2t}'(1))\right)$$

$$= \lim_{n\to\infty} E\left(n^{-1}\alpha'\Sigma^{-1}\sum_{t=1}^{n}(E(u_tu_t'|\mathcal{F}_{t-1})\Sigma^{-1/2} \otimes R_{2t}'(1))\right)$$

$$= \alpha'\Sigma^{-1}(\Sigma^{1/2} \otimes E(R_{20}'(1))) = 0,$$

where $E(R_{20}(1)) = E(w_0') - E(w_0z_0')E(z_0z_0')^{-1}E(z_0)$.

In the same way, we can show $E(W_2(s)K(s)') = 0$.

Proof of Theorem 2. If we denote $R_{0t}(\tau) = \Delta x_t - \sum_{t=1}^{[n\tau]}\Delta x_t z_{t-1}'(\sum_{t=1}^{[n\tau]} z_{t-1}z_{t-1}')^{-1} z_{t-1}$, then the model can be written

$$R_{0t}(\tau) = \alpha\gamma' R_{1t}(\tau) + u_t(\tau),$$

where $u_t(\tau) = u_t - \sum_{t=1}^{[n\tau]} u_t z_{t-1}'(\sum_{t=1}^{[n\tau]} z_{t-1}z_{t-1}')^{-1} z_{t-1}$.

$$n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)\tilde{v}_t' \Rightarrow \int_0^\tau W_2\,dB_1' - \int_0^\tau W_2W_2'\left(\int_0^1 W_2W_2'\right)^{-1}\int_0^1 W_2\,dB_1'. \quad \textbf{(A.5)}$$

The estimated residual satisfies

$$\tilde{u}_t = u_t - (\tilde{\alpha} - \alpha)\tilde{\gamma}' R_{1t}(\tau) - \alpha(\tilde{\beta} - \beta)R_{12t}(\tau) - \sum_{t=1}^{[n\tau]} u_t z_{t-1}'\left(\sum_{t=1}^{[n\tau]} z_{t-1}z_{t-1}'\right)^{-1} z_{t-1}.$$

Because

$$S_{12n}(\tau) = n^{-1}\sum_{t=1}^{[n\tau]} x_{2t-1}w_{t-1} - n^{-1}\sum_{t=1}^{[n\tau]} x_{2t-1}z'_{t-1}\left(n^{-1}\sum_{t=1}^{[n\tau]} z_{t-1}z'_{t-1}\right)^{-1}$$

$$n^{-1}\sum_{t=1}^{[n\tau]} z_{t-1}w'_{t-1} + n^{-1}S_{11n}(\tau)n(\tilde{\beta}-\beta),$$

we can show $S_{12n}(\tau) = O_{p\tau}(1)$ by using (A.6) and Lemma 9.

$$\begin{aligned} n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)\tilde{u}'_t &= n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)u'_t - n^{-1}S_{11n}(\tau)n(\tilde{\beta}-\beta)\alpha' \\ &\quad - n^{-1/2}S_{12n}(\tau)\sqrt{n}(\tilde{\alpha}-\alpha)' \\ &= n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)u'_t - n^{-1}S_{11n}(\tau)n(\tilde{\beta}-\beta)\alpha' + o_{p\tau}(1) \\ &\Rightarrow \int_0^\tau W_2\,dW' - \int_0^\tau W_2W_2'\left(\int_0^1 W_2W_2'\right)^{-1} \\ &\quad \int_0^1 W_2\,dW'\Sigma^{-1}\alpha(\alpha'\Sigma^{-1}\alpha)^{-1}\alpha', \end{aligned}$$

$$\begin{aligned} n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)\tilde{v}'_t &= n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}(\tau)\tilde{u}'_t\tilde{\Sigma}^{-1}\tilde{\alpha}'(\tilde{\alpha}'\tilde{\Sigma}^{-1}\tilde{\alpha})^{-1/2} \\ &\Rightarrow \int_0^\tau W_2\,dB_1' - \int_0^\tau W_2W_2'\left(\int_0^1 W_2W_2'\right)^{-1}\int_0^1 W_2\,dB_1' \end{aligned}$$

because $\tilde{\alpha} - \alpha = o_p(1)$ and $\tilde{\Sigma} - \Sigma = o_p(1)$.

Because $n^{-1}S_{11n}(\tau) \Rightarrow \int_0^\tau W_2W_2'$ by Lemma 2,

$$V_{11n}(\tau) \Rightarrow \int_0^\tau W_2W_2' - \int_0^\tau W_2W_2'\left(\int_0^1 W_2W_2'\right)^{-1}\int_0^\tau W_2W_2'. \tag{A.6}$$

Because $(C_2(1)\Sigma C_2'(1))^{1/2}$ is nonsingular, $\mathrm{LM}_n^\beta(\tau) \Rightarrow \mathrm{LM}_1^\beta(\tau)$ by the CMT. ■

Proof of Theorem 3. Let $e_t(\tau) = \Sigma^{-1/2'}u_t \otimes R_{2t}(\tau)$.

$$n^{-1/2}\sum_{t=1}^{[n\tau]} e_t(\tau) \Rightarrow K(\tau) = BM(I \otimes \tau Q). \tag{A.7}$$

Let $e_{1t} = \Sigma^{-1/2'}u_t \otimes w_{t-1}$ and $e_{2t} = \Sigma^{-1/2'}u_t \otimes z_{t-1}$.

Because $\{e_{1t}|\mathcal{F}_t\}$ and $\{e_{2t}|\mathcal{F}_t\}$ are martingale difference sequences, and $E|\check{z}_0|^2 < \infty$,

$$\begin{pmatrix} n^{-1/2}\sum_{t=1}^{[n\tau]} e_{1t} \\ n^{-1/2}\sum_{t=1}^{[n\tau]} e_{2t} \end{pmatrix} \Rightarrow \begin{pmatrix} K_1(\tau) \\ K_2(\tau) \end{pmatrix} = BM(I \otimes \tau E(\check{z}_0\check{z}_0')).$$

Because $e_t(\tau) = e_{1t} - [I \otimes \sum_{t=1}^{[n\tau]} w_{t-1} z'_{t-1} (\sum_{t=1}^{[n\tau]} z_{t-1} z'_{t-1})^{-1}] e_{2t}$,

$$n^{-1/2} \sum_{t=1}^{[n\tau]} e_t(\tau) \Rightarrow K_1(\tau) - [I \otimes E(w_0 z'_0) E(z_0 z'_0)^{-1}] K_2(\tau) \equiv K(\tau)$$
$$= BM(I \otimes \tau Q).$$

$$n^{-1/2} \sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau) u'_t = n^{-1/2} \sum_{t=1}^{[n\tau]} R_{2t}(\tau) u'_t + o_{p\tau}(1). \quad \textbf{(A.8)}$$

Because $\tilde{w}_t = w_t + (\tilde{\beta} - \beta)' x_{2t}$,

$$n^{-1/2} \sum_{t=1}^{[n\tau]} \tilde{w}_{t-1} u'_t = n^{-1/2} \sum_{t=1}^{[n\tau]} w_{t-1} u'_t + n^{-1/2} n(\tilde{\beta} - \beta)' n^{-1} \sum_{t=1}^{[n\tau]} x_{2t-1} u'_t$$
$$= n^{-1/2} \sum_{t=1}^{[n\tau]} w_{t-1} u'_t + o_{p\tau}(1),$$

$$n^{-1} \sum_{t=1}^{[n\tau]} \tilde{w}_{t-1} z'_{t-1} = n^{-1} \sum_{t=1}^{[n\tau]} w_{t-1} z'_{t-1} - n^{-1} n(\tilde{\beta} - \beta)' n^{-1} \sum_{t=1}^{[n\tau]} x_{2t-1} z'_{t-1}$$
$$= n^{-1} \sum_{t=1}^{[n\tau]} w_{t-1} z'_{t-1} + o_{p\tau}(1).$$

$$S_{22n}(\tau) = n^{-1} \sum_{t=1}^{[n\tau]} R_{2t}(\tau) R'_{2t}(\tau) + o_{p\tau}(1). \quad \textbf{(A.9)}$$

$$\tilde{R}_{2t}(\tau) = R_{2t}(\tau) + n(\tilde{\beta} - \beta)' n^{-1} x_{2t-1} - n(\tilde{\beta} - \beta)' n^{-1}$$
$$\sum_{t=1}^{[n\tau]} x_{2t-1} z'_{t-1} \left(n^{-1} \sum_{t=1}^{[n\tau]} z_{t-1} z'_{t-1} \right)^{-1} n^{-1} z_{t-1}.$$

$$n^{-1/2} \sum_{t=1}^{[n\tau]} (\tilde{\Sigma}^{-1/2\prime} \tilde{u}_t \otimes \tilde{R}_{2t}(\tau)) \Rightarrow K(\tau) - \tau K(1). \quad \textbf{(A.10)}$$

$$n^{-1/2} \sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau) \tilde{u}'_t = n^{-1/2} \sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau) u'_t - S_{22n}(\tau) \sqrt{n} (\tilde{\alpha} - \alpha)'$$
$$- n^{-1/2} S_{21n}(\tau) n(\tilde{\beta} - \beta) \alpha'$$
$$= n^{-1/2} \sum_{t=1}^{[n\tau]} \tilde{R}_{2t}(\tau) u'_t - S_{22n}(\tau) \sqrt{n} (\tilde{\alpha} - \alpha)' + o_{p\tau}(1),$$

so

$$n^{-1/2} \sum_{t=1}^{[n\tau]} (\tilde{\Sigma}^{-1/2\prime} \tilde{u}_t \otimes \tilde{R}_{2t}(\tau)) = n^{-1/2} \sum_{t=1}^{[n\tau]} (\Sigma^{-1/2\prime} u_t \otimes R_{2t}(\tau))$$
$$- \left(\Sigma^{-1/2\prime} \otimes n^{-1} \sum_{t=1}^{[n\tau]} R_{2t}(\tau) R'_{2t}(\tau) \right) \sqrt{n}\, \mathrm{vec}(\tilde{\alpha} - \alpha)'$$
$$+ o_{p\tau}(1)$$
$$\Rightarrow K(\tau) - (\Sigma^{-1/2\prime} \otimes \tau Q)(\Sigma^{1/2} \otimes Q^{-1}) K(1)$$
$$= K(\tau) - \tau K(1).$$

Because $V_{22n}(\tau) = S_{22n}(\tau) - S_{22n}(\tau)S_{22n}(1)^{-1}S_{22n}(\tau) \to^p \tau(1-\tau)Q$ uniformly in $\tau \in \tau^*$,

$$(I \otimes V_{22n}(\tau))^{-1/2'} n^{-1/2} \sum_{t=1}^{[n\tau]} (\tilde{\Sigma}^{-1/2'} \tilde{u}_t \otimes \tilde{R}_{2t-1}(\tau)) \Rightarrow (\tau(1-\tau))^{-1/2}[J(\tau) - \tau J(1)].$$

Thus, $\mathrm{LM}_n^{\alpha}(\tau) \Rightarrow 1/\tau(1-\tau)\,(J(\tau) - \tau J(1))'(J(\tau) - \tau J(1))$. ■

Proof of Lemma 3. The representation theorem in the model with deterministic trends is

$$\Delta x_t = C(L)(\mu + u_t),$$

$$x_t = \mu_0 + C(1)\sum_{i=1}^{t} u_i + \Phi(L)u_t \quad \text{if } \alpha_\perp' \mu = 0,$$

and

$$x_t = \mu_0 + C(1)\mu t + C(1)\sum_{i=1}^{t} u_i + \Phi(L)u_t \quad \text{if } \alpha_\perp' \mu \neq 0.$$

Using Lemmas 8 and 9,

$$\begin{aligned}
n^{-1/2}R_{1t}^*(\tau) &= n^{-1/2}x_{t-1} - n^{-1/2}\sum_{t=1}^{[n\tau]} x_{t-1}z_{t-1}^{*\prime}\left(\sum_{t=1}^{[n\tau]} z_{t-1}^* z_{t-1}^{*\prime}\right)^{-1} z_{t-1}^* \\
&= n^{-1/2}x_{t-1} - n^{-1/2}\sum_{t=1}^{[n\tau]} x_{t-1}z_{t-1}^{*\prime}\delta_n\left(\sum_{t=1}^{[n\tau]} \delta_n z_{t-1}^* z_{t-1}^{*\prime}\delta_n\right)^{-1} \delta_n z_{t-1}^* \\
&= n^{-1/2}x_{t-1} - n^{-1/2}\frac{1}{[n\tau]}\sum_{t=1}^{[n\tau]} x_{t-1} + o_{p\tau}(1) \\
&= n^{-1/2}x_{t-1}^{\circ} - n^{-1/2}\frac{1}{[n\tau]}\sum_{t=1}^{[n\tau]} x_{t-1}^{\circ} + o_{p\tau}(1),
\end{aligned}$$

where $\delta_n^{-1} = \mathrm{diag}(1,\sqrt{n},\sqrt{n},\dots)$, and $x_t^{\circ} = C(1)\sum_{i=1}^{t} u_t$.
Thus, $n^{-1/2}R_{12[ns]}^*(\tau) \Rightarrow C_2(1)W - 1/\tau C_2(1)\int_0^\tau W$. ■

Proof of Lemma 6. Since demeaning cannot remove a linear trend in the DGP, the projected series contains the linear trend. If we use Lemmas 8 and 9, then

$$n^{-1/2}R_{1t}^*(\tau) = n^{-1/2}x_{t-1}^{\circ} - n^{-1/2}\frac{1}{[n\tau]}\sum_{t=1}^{[n\tau]} x_t^{\circ} + \sqrt{n}\nu(t/n - \tau/2) + o_{p\tau}(1).$$

Thus,

$$n^{-1/2}\nu_\perp' R_{12[ns]}^*(\tau) \Rightarrow \nu_\perp' C_2(1)W^*(s,\tau), \quad \text{and} \quad n^{-1}\nu' R_{12[ns]}^*(\tau) \Rightarrow \nu'\nu(s - \tau/2).$$ ■

Proof of Theorem 4. If we denote $R_{0t}^*(\tau) = \Delta x_t - \sum_{t=1}^{[n\tau]} \Delta x_t z_{t-1}^{*\prime}(\sum_{t=1}^{[n\tau]} z_{t-1}^* z_{t-1}^{*\prime})^{-1} z_{t-1}^*$, then the model can be written

$$R_{0t}^*(\tau) = \alpha\gamma' R_{1t}^*(\tau) + u_t(\tau),$$

where $u_t(\tau) = u_t - \sum_{t=1}^{[n\tau]} u_t z_{t-1}^{*\prime}(\sum_{t=1}^{[n\tau]} z_{t-1}^{*} z_{t-1}^{*\prime})^{-1} z_{t-1}^{*}$. The estimated residual satisfies

$$\tilde{u}_t = u_t - (\tilde{\alpha} - \alpha)\tilde{\gamma}' R_{1t}^{*}(\tau) - \alpha(\tilde{\beta} - \beta)' R_{12t}^{*}(\tau) - \sum_{t=1}^{[n\tau]} u_t z_{t-1}^{*\prime}\left(\sum_{t=1}^{[n\tau]} z_{t-1}^{*} z_{t-1}^{*\prime}\right)^{-1} z_{t-1}^{*}.$$

$$\begin{aligned} n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}^{*}(\tau)\tilde{u}_t' &= n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}^{*}(\tau)u_t' - n^{-1}S_{11n}^{*}(\tau)n(\tilde{\beta} - \beta)\alpha' \\ &\quad - n^{-1/2}S_{12n}^{*}(\tau)\sqrt{n}(\tilde{\alpha} - \alpha)' \\ &= n^{-1/2}\sum_{t=1}^{[n\tau]} R_{12t}^{*}(\tau)u_t' - n^{-1}S_{11n}^{*}(\tau)n(\tilde{\beta} - \beta)\alpha' + o_{p\tau}(1) \end{aligned}$$

because $S_{12n}^{*}(\tau) = O_{p\tau}(1)$, and $\sqrt{n}(\hat{\alpha} - \alpha)' = O_p(1)$.

$$\begin{aligned} n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}^{*}(\tau)\tilde{v}_t' &= n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}^{*}(\tau)v_t' - n^{-1}S_{11n}^{*}(\tau)n(\tilde{\beta} - \beta)(\alpha'\Sigma^{-1}\alpha)^{1/2} + o_{p\tau}(1) \\ &= n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}^{*}(\tau)v_t' - n^{-1}S_{11n}^{*}(\tau)(n^{-1}S_{11n}^{*}(1))^{-1}n^{-1}\sum_{t=1}^{n} R_{12t}^{*}(1)v_t' \\ &\quad + o_{p\tau}(1). \end{aligned}$$

We define $A = (\nu_{\perp}, n^{-1/2}\nu)$ and $D = \mathrm{diag}((\nu_{\perp}' C_2(1)\Sigma C_2'(1)\nu_{\perp})^{1/2}, \nu'\nu)$. By using Lemma 6, we have the following results:

$$A' n^{-1}\sum_{t=1}^{[n\tau]} R_{12t}^{*}(\tau)v_t' \Rightarrow D' \int_0^{\tau} \bar{B}_2(s,\tau)\, dB_1'(s),$$

and

$$n^{-2}A'\sum_{t=1}^{[n\tau]} R_{12t}^{*}(\tau)R_{12t}^{*\prime}(\tau)A \Rightarrow D' \int_0^{\tau} \bar{B}_2(s,\tau)\bar{B}_2'(s,\tau)\, ds\, D.$$

Because D is nonsingular, $\mathrm{LM}_n^{\beta *}(\tau) \Rightarrow \mathrm{LM}_3^{\beta}(\tau)$ by the CMT. ■

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